

# Geometry of prime end boundary and the Dirichlet problem for bounded domains in metric measure spaces

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**Abstract.** In this note we study the Dirichlet problem associated with a version of prime end boundary of a bounded domain in a complete metric measure space equipped with a doubling measure supporting a Poincaré inequality. We show the resolvitivity of functions that are continuous on the prime end boundary and are Lipschitz regular when restricted to the subset of all prime ends whose impressions are singleton sets. We also consider a new notion of capacity adapted to the prime end boundary, and show that bounded perturbations of such functions on subsets of the prime end boundary with zero capacity are resolvable and that their Perron solutions coincide with the Perron solution of the original functions. We also describe some examples which demonstrate the efficacy of the prime end boundary approach in obtaining new results even for the classical Dirichlet problem for some Euclidean domains.

*Key words and phrases:* Prime end boundary, Dirichlet problem,  $p$ -harmonic functions, Perron method, metric measure spaces, doubling measure, Poincaré inequality.

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## 1. Introduction

The Dirichlet problem associated with a partial differential operator  $L$  on a domain  $\Omega$  is the problem of finding a function  $u$  on  $\Omega$  such that  $Lu = 0$  (usually in a weak sense) on  $\Omega$  and  $u - f \in W_0^{1,p}(\Omega)$  for a given boundary data  $f : \partial\Omega \rightarrow \mathbb{R}$ . However, in some situations  $\partial\Omega$  is not the correct boundary to be considered. For example, given a flat metal disc, if we cut a radial slit in the disc and insert a non-conducting material in the slit, then heat energy cannot pass from one side of the slit to the other directly, and so in this case the correct boundary for the slit disc (when the operator  $L$  is the one associated with the heat equation) should count each point on the slit twice, once for each side of the slit. For more complicated domains the corresponding natural boundary is more complicated. To address this issue, the paper [1] proposed an alternative for the topological boundary  $\partial\Omega$ , called the prime end boundary. The goal of this note is to use the prime end boundary in the study of the Dirichlet problem.

In this note we consider a variational analog of the  $p$ -Laplacian  $\Delta_p$  in the setting of bounded domains in complete metric measure spaces equipped with a doubling

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measure supporting a  $p$ -Poincaré inequality. We use the Perron method to construct solutions to the Dirichlet problem on bounded domains in such metric spaces.

The Perron method was successfully used in [4] to construct solutions to the Dirichlet problem in the metric setting when the boundary considered is the topological boundary. We demonstrate in this paper that such a method also works for the prime end boundary. The paper [6] considered the Dirichlet problem for the prime end boundary in the simple situation that each prime end has only one point in its impression (see Section 2 for the definitions of these concepts) and that the prime end boundary is compact. However, in general the prime end boundary is not compact, as even the simple example of the harmonic comb shows (see the examples in Section 8). Hence the principal part of the work of this note is to overcome the non-compactness issue of the prime end boundary in applying the Perron method.

The standard assumption in this paper is that the metric space under study is a complete metric space equipped with a doubling measure supporting a  $p$ -Poincaré inequality for some fixed  $1 < p < \infty$ . We use the Newtonian spaces as substitutes for Sobolev spaces under this assumption quite successfully, but we point out that an alternate construction of Sobolev-type spaces has been used by others successfully in some fractal spaces that do not satisfy the Poincaré inequality considered here; see for example [8] and the references therein. The paper [17] considers the Dirichlet problem associated with the  $p$ -Laplacian on domains in metric measure spaces that satisfy the doubling and Poincaré inequality assumptions; however, the boundary that paper studies is the  $p$ -Royden algebra boundary, which is a functional analytic construct. The notion of prime end boundary we consider in this paper is more of a geometric construct.

The structure of this paper is as follows. In Section 2 we explain the notation used in this paper, in particular we explain the construction of the prime end boundary. One should keep in mind that even in the setting of simply connected planar domains the prime end boundary described in Section 2 could differ from that of Carathéodory (see [1]), but it has the advantage of being usable for non-simply connected planar domains and more general domains in higher dimensional Euclidean spaces as well as domains in metric measure spaces. In Section 3 we continue the explanation of concepts used by describing the analog of Sobolev spaces in the metric setting, called the Newtonian spaces, and by describing the relevant associated potential theory. In Section 4 we explore some structures associated with the prime end boundaries, and in Assumption 4.7 we give a natural condition on the domain needed in the rest of the paper. Many domains whose prime end boundaries are not compact do satisfy this condition (see the examples in Section 8 for a sampling), but we do not know of any domain that would violate this condition.

In Section 5 we gather some additional properties of prime end boundaries of domains that satisfy Assumption 4.7, including the key property that if the boundary of a connected open subset of  $\Omega$  intersects the topological boundary of  $\Omega$ , then it must, under the prime end closure topology, intersect the prime end boundary of  $\Omega$ ; see Theorem 5.3. In Section 6 we propose a modification of the  $p$ -capacity used in [6], adapted to the prime end boundary, and study its basic capacitary properties. We also show in this section that functions in the Newtonian class of the domain with zero boundary values (denoted  $N_0^{1,p}(\Omega)$ ) are quasicontinuous with respect to this new capacity. In Section 7 we use the above notions together with the Perron method, adapted to the prime end boundary, to obtain resolutivity properties of certain continuous functions on the prime end boundary of  $\Omega$ . We also show stability of the Perron solution under bounded perturbation of these functions on sets of (new) capacity zero. Finally, in Section 8 we describe three examples and use them to show how, even in the Euclidean setting, new stability results for the classical Dirichlet problem can be obtained from the prime end boundary approach.

## 2. Preliminaries: the prime end boundary

In this paper we assume that  $(X, d)$  is a complete, doubling metric space that is quasiconvex. Recall that  $X$  is quasiconvex if there is a constant  $C_q \geq 1$  such that whenever  $x, y \in X$ , there is a rectifiable curve (that is, a curve of finite length)  $\gamma$  with end points  $x$  and  $y$  such that the length of  $\gamma$ , denoted  $\ell(\gamma)$ , is at most  $C_q d(x, y)$ . Quasiconvexity is a consequence of the validity of a  $p$ -Poincaré inequality on the metric measure space  $(X, d, \mu)$  when  $\mu$  is doubling, and from Section 6 onward we will assume that  $\mu$  is doubling and supports a  $p$ -Poincaré inequality. So the assumption of quasiconvexity here is not overly constrictive. Furthermore, complete doubling metric spaces have a highly useful topological property called properness. A metric space is proper if closed and bounded subsets of the space are compact. This property will enable us to apply the Arzela-Ascoli theorem in subsequent sections of this paper. To see that a complete metric space  $X$  equipped with a doubling measure is proper, we may argue as follows. Since  $X$  is equipped with a doubling measure, it is doubling in the sense of [12, Section 10.13], see [12, page 82]. It follows that closed balls in such a space  $X$  are complete and totally bounded, and so are compact; see [19, page 275, Theorem 3.1].

We essentially follow [1] in the construction of prime ends for bounded domains in  $X$ . In what follows,  $\Omega \subset X$  is a bounded open connected set.

In addition to the standard metric balls  $B(x, r) := \{y \in X : d(x, y) < r\}$ , we will also make use of the  $r$ -neighborhood of a set, defined as

$$N(A, r) := \bigcup_{x \in A} B(x, r). \quad (2.1)$$

We will also use the notion of the distance from a point to a set and distance between two sets:

$$\text{dist}(x, A) := \inf\{d(x, y) : y \in A\}, \quad \text{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}.$$

Since  $X$  is quasiconvex, it is easy to see by a topological argument that an open connected subset of  $X$  is rectifiably connected. A proof of this appears in [3, Lemma 4.38], but since the proof is elementary, we also give a proof here. Indeed, given an open connected subset  $U$  of  $X$  and  $x \in U$ , consider the collection  $U(x)$  of all points  $y \in U$  such that there is a rectifiable curve in  $U$  connecting  $x$  to  $y$ . Quasiconvexity of  $X$  implies that whenever  $y \in U(x)$ , there is a ball centered at  $y$  contained in  $U(x)$ . Thus  $U(x)$  is an open subset of  $U$ . Similar argument gives  $U \setminus U(x)$  is also open, and since  $U$  is connected, this means that either  $U(x)$  is empty or  $U(x) = U$ . Because  $x \in U(x)$ , it follows that  $U(x) = U$ , and so  $U$  is rectifiably connected.

**Definition 2.1.** Given a set  $U \subset X$ , the *inner distance* on  $U$  is given for  $x, y \in U$  by

$$d_{\text{inn}}^U(x, y) = \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken over rectifiable curves  $\gamma$  in  $U$  with end points  $x, y$ .

If  $U$  is not connected and  $x, y$  belong to different components of  $U$ , then we have  $d_{\text{inn}}^U(x, y) = \infty$ . However, if  $U$  is a connected open subset of  $X$ , then, by the comments before the above definition, we know that  $d_{\text{inn}}^U$  is a metric on  $U$ . Given that  $X$  is complete and proper, an application of the Arzelà-Ascoli theorem tells us that if  $d_{\text{inn}}^U(x, y)$  is finite, then there is a  $d_{\text{inn}}^U$ -geodesic  $\gamma_{x,y}^U$  connecting  $x$  to  $y$  in  $\overline{U}$  with length  $\ell(\gamma_{x,y}^U) = d_{\text{inn}}^U(x, y)$ . Here, by a  $d_{\text{inn}}^U$ -geodesic we mean a curve in  $\overline{U}$  connecting  $x$  to  $y$  that appears as a uniform limit of a sequence of length-minimizing curves in  $U$  connecting  $x$  to  $y$ . Furthermore, the quasiconvexity of  $X$  implies that,

if  $U$  is open, then, for each  $x \in U$  with  $r = \text{dist}(x, X \setminus U)/C_q$ , the two metrics  $d$  and  $d_{\text{inn}}^U$  are biLipschitz equivalent on  $B(x, r)$  with biLipschitz constant  $C_q$ .

We will make use of the Mazurkiewicz distance, defined below.

**Definition 2.2.** Let  $\Omega$  be a bounded connected open subset of  $X$ , that is,  $\Omega$  is a bounded domain. Given  $x, y \in \Omega$ , the *Mazurkiewicz distance*  $d_M$  between  $x$  and  $y$  on  $\Omega$  is

$$d_M(x, y) = \inf_E \text{diam } E,$$

where the infimum is taken over all connected sets  $E \subset \Omega$  with  $x, y \in E$ .

It is clear that  $d_M$  is a metric on  $\Omega$ , with  $d(x, y) \leq d_M(x, y) \leq d_{\text{inn}}^\Omega(x, y)$ . The completion of  $\Omega$  under  $d_M$  is denoted  $\overline{\Omega}^M$ , with  $\partial_M \Omega := \overline{\Omega}^M \setminus \Omega$ . The metric  $d_M$  extends naturally to a metric on  $\overline{\Omega}^M$ ; this extended metric will also be denoted by  $d_M$ .

Note that (2.1) can be applied to the distances in Definitions 2.1 and 2.2, with the new  $r$ -neighborhoods being denoted  $N_{\text{inn}}^U(x, r)$  and  $N_M(x, r)$  respectively.

**Definition 2.3.** A set  $E \subset \Omega$  is *acceptable* if  $E$  is connected and  $\overline{E} \cap \partial \Omega$  is non-empty. A sequence  $\{E_k\}_{k \in \mathbb{N}}$  of acceptable sets is a *chain* if all of the following conditions hold true:

- (a)  $E_{k+1} \subset E_k$  for  $k \in \mathbb{N}$ ,
- (b) for each  $k \in \mathbb{N}$ , the distance  $\text{dist}_M(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) > 0$ ,
- (c) the *impression*  $I(\{E_k\}_k) := \bigcap_{k \in \mathbb{N}} \overline{E_k}$  is a subset of  $\partial \Omega$ .

Note that  $I(\{E_k\}_k)$  is a compact, connected set.

Our definition differs slightly from that given in [1] in that condition (b) now references the Mazurkiewicz distance. However, the examples and results of [1] still hold. Indeed, whenever the analog of condition (b) was used in [1] to prove a claim, the key property used was that when  $\{E_k\}_{k \in \mathbb{N}}$  is a chain, for each  $k$  and points  $x \in E_{k+1}$  and  $y \in \Omega \setminus E_k$ , every connected compact subset of  $\Omega$  that contains both  $x$  and  $y$  must have diameter bounded below by a positive number that may depend on  $k$  but not on  $x, y$ . This is precisely the condition given by *our* version of condition (b), and so the results of [1] hold for our ends as well. The principal result of [1] we depend on is the identification of ends that have singleton impressions as certain prime ends. For the convenience of the reader we will give a proof of that fact here; see Lemma 2.6 below. The examples given in [1] are simple enough that it can be directly verified that the prime ends for those example domains in the sense of [1] are the same as those in our sense. While we have more chains than [1] the additional chains are equivalent (in the sense described in Definition 2.4 below) to chains that satisfy the conditions of [1].

It is easy to see that a chain in the sense of [1] is a chain in our sense, but the converse need not be true. Therefore in general we have more chains in the sense of Definition 2.3 than does [1]. Therefore conceivably we have more ends than does [1] and thus an end that might be prime in the setting of [1] (see the definition of prime ends below) may not be prime in our sense. However, given that the notion of Sobolev spaces in the metric setting considered here uses paths extensively, the Mazurkiewicz distance seems to be the natural one to consider here. We point out that we are in good company here; it was shown by Näkki that condition (b) is equivalent to an Ahlfors-type condition regarding extremal length when the domain is a quasiconformally collared Euclidean domain, see [20].

We have also chosen to use the Mazurkiewicz distance  $\text{dist}_M$  rather than the original metric distance  $d$  (as [1] does) because in constructing ends that intersect certain open subsets of  $\Omega$ , it is easier to describe the construction when  $\text{dist}_M$  is used rather than  $\text{dist}$ ; see Section 5. Thus the use of  $\text{dist}_M$  makes for a simpler

exposition, and hence we have chosen to give the above modification. We again point out that whenever the analog of condition (b) was used in a proof in [1], it is actually the positivity of  $\text{dist}_M$ -distance that was needed. Hence we do not lose anything by our modification.

**Definition 2.4.** Given two chains  $\{E_k\}_k$  and  $\{F_k\}_k$ , we say that  $\{E_k\}_k$  *divides*  $\{F_k\}_k$  if, for each positive integer  $k$ , there is a positive integer  $j_k$  such that  $E_{j_k} \subset F_k$ .

The above notion of division gives an equivalence relationship on the collection of all chains; two chains  $\{E_k\}_k$  and  $\{F_k\}_k$  are equivalent if they both divide each other. Given a chain  $\{E_k\}_k$ , its equivalence class is denoted  $[\{E_k\}_k]$ . If two chains  $\{E_k\}_k$  and  $\{F_k\}_k$  are equivalent, then their impressions are equal. Let this (common) impression be denoted  $I[\{E_k\}_k]$ . These equivalence classes are called *ends* of  $\Omega$ . The collection of all ends of  $\Omega$  is called the *end boundary*  $\partial_E \Omega$  of  $\Omega$ .

Observe also that if a chain  $\{E_k\}_k$  divides another chain  $\{G_k\}_k$ , and  $\{F_k\}_k \in [\{E_k\}_k]$ , then  $\{F_k\}_k$  also divides  $\{G_k\}_k$ . Furthermore,  $\{E_k\}_k$  divides every chain in  $[\{G_k\}_k]$ . Hence the notion of divisibility extends to ends as well. We take as notation  $[\{E_k\}_k] \mid [\{G_k\}_k]$  to mean that  $[\{E_k\}_k]$  divides  $[\{G_k\}_k]$ .

**Definition 2.5.** An end of  $\Omega$  is a *prime end* if the only end that divides it is itself. The collection of all prime ends of  $\Omega$ , called the *prime end boundary of  $\Omega$* , is denoted  $\partial_P \Omega$ . The collection of all prime ends of  $\Omega$  with singleton impression is called the *singleton prime end boundary* and is denoted  $\partial_{SP} \Omega$ .

**Lemma 2.6.** Let  $\{E_k\}_k$  be a chain such that  $I(\{E_k\}_k) = \{x_0\}$ , that is, the chain has only a singleton impression. Then  $[\{E_k\}_k]$  is a prime end, and for each positive integer  $k$  there is a positive real number  $r_k > 0$  such that a connected component of  $B(x, r_k) \cap \Omega \subset E_k$ .

*Proof.* We will prove the second part of the lemma, for then the first part follows from [1, Lemma 7.3] and [1, Corollary 7.11] (see also the discussion in [1, Section 10] and [7]).

Suppose that there is no such positive number  $r_k$ . Then for each  $r > 0$  let  $F_k(r)$  be the connected component of  $B(x_0, r) \cap \Omega$  containing points  $x_k^r \in E_{k+1}$ . Since  $F_k(r) \not\subset E_k$ , it follows that there is a point  $y_k^r \in F_k(r) \setminus E_k$ . Because  $F_k(r)$  is a connected open subset of the quasi convex space  $X$ , it follows that  $F_k(r)$  is rectifiably connected (see the discussion before Definition 2.1). Thus there is a compact curve  $\gamma$  in  $F_k(r)$  connecting  $x_k^r \in E_{k+1}$  to  $y_k^r \notin E_k$ , and the diameter of such a curve is at most  $2r$ . Thus

$$0 < \text{dist}_M(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) \leq \text{diam}(\gamma) \leq 2r,$$

the above inequality holding for each  $r > 0$ . This is not possible. Hence such a positive number  $r_k$  must exist.  $\square$

The following series of definitions describes a topology on  $\partial_E \Omega$  that meshes well with the topology of  $\Omega$ . We first “stitch”  $\partial_E \Omega$  to  $\Omega$  via a sequential topology as follows.

**Definition 2.7.** Given a sequence  $\{x_i\}_i$  in  $\Omega$ , we say that  $x_i \xrightarrow{\overline{\Omega}^P} [\{E_k\}_k]$  if for every positive integer  $k$  there is a positive integer  $i_k$  such that whenever  $i \geq i_k$  we have  $x_i \in E_k$ .

One should be aware that a sequence in  $\Omega$  can converge to two *different* ends, as [1, Example 8.9] shows.

We next extend the topology to  $\partial_E \Omega$  by describing sequential topology on  $\partial_E \Omega$ .

**Definition 2.8.** Given a sequence  $\{[E_k^n]_k\}_n$  of ends of  $\Omega$  and an end  $[E_k^\infty]_k$  of  $\Omega$ , we say that  $[E_k^n]_k \xrightarrow{\overline{\Omega}^P} [E_k^\infty]_k$  if for each positive integer  $k$  there is a positive integer  $n_k$  such that whenever  $n \geq n_k$ , there is a positive integer  $j_n$  such that  $E_{j_n}^n \subset E_k^\infty$ .

A modification of [1, Example 8.9] shows that a sequence of ends can converge to more than one end. However, a sequence of ends will never converge to a point in  $\Omega$ .

**Definition 2.9.** Equip the set  $\overline{\Omega}^E := \Omega \cup \partial_E \Omega$  with the sequential topology associated with the above notion of limits. Equip the subset  $\overline{\Omega}^P := \Omega \cup \partial_P \Omega$  with the subspace topology inherited from  $\overline{\Omega}^E$ . We call the sets  $\overline{\Omega}^E$  and  $\overline{\Omega}^P$  the *End Closure* of  $\Omega$  and the *Prime End Closure* of  $\Omega$  respectively.

Sometimes, it may be useful to talk about the closure or boundary of a set  $V \subset \overline{\Omega}^P$  with respect to the Prime End topology of  $\Omega$ . To avoid confusion we will denote the *Prime End closure of  $V$  with respect to the Prime End topology on  $\Omega$*  as  $\overline{V}^{P,\Omega}$  and the *Prime End boundary of  $V$  with respect to the Prime End topology of  $\Omega$*  as  $\partial_P^\Omega V$ . Note that if  $\overline{V} \subset \Omega$ , then  $\overline{V}^{P,\Omega} = \overline{V}$  and  $\partial_P^\Omega V = \partial V$ .

**Remark 2.10.** Recall that by  $\partial_{SP} \Omega$  we mean the collection of all prime ends of  $\Omega$  whose impressions contain only one point. Recall the Mazurkiewicz boundary  $\partial_M \Omega$  of  $\Omega$  from Definition 2.2. Though  $\overline{\Omega}^P$  admits no metric, it is shown in [1, Theorem 9.5] that there is a homeomorphism  $\Phi : \Omega \cup \partial_{SP} \Omega \rightarrow \overline{\Omega}^M$  such that  $\Phi|_\Omega$  is the identity map and  $\Phi|_{\partial_{SP} \Omega} : \partial_{SP} \Omega \rightarrow \partial_M \Omega$ . It follows that  $\Omega \cup \partial_{SP} \Omega$  is metrizable via the pullback of the metric  $d_M$ . So, for  $x, y \in \Omega \cup \partial_{SP} \Omega$ , by  $d_M(x, y)$  we truly mean  $d_M(\Phi(x), \Phi(y))$ .

**Remark 2.11.** Given a set  $G \subset \Omega$ , we define

$$G^P := G \cup \{[E_k]_k \in \partial_P \Omega \mid \text{for some } j, E_j \subset G\}.$$

It was shown in [1, Proposition 8.5] that the collection of sets

$$\{G, G^P \mid G \subset \Omega \text{ is open}\}$$

forms a basis for the topology on  $\overline{\Omega}^P$ . Note that given the above definition of  $G^P$ , we have  $\overline{\Omega}^P = \Omega^P$ . In the next few sections, we will focus on the sequential definition of this topology. In later sections, the above natural basis will prove invaluable in making our results more intuitive.

**Definition 2.12.** We say that a point  $x_0 \in \partial \Omega$  is *accessible from  $\Omega$*  if there is a curve  $\gamma : [0, 1] \rightarrow \overline{\Omega}$  such that  $\gamma(1) = x_0$  and  $\gamma([0, 1)) \subset \Omega$ . We say that a point  $x_0 \in \partial \Omega$  is *accessible through the chain  $\{E_k\}_k$*  if there is such a curve  $\gamma$  satisfying in addition that for each positive integer  $k$  there is some  $0 < t_k < 1$  with  $\gamma([t_k, 1)) \subset E_k$ . The curve  $\gamma$  is said to *access  $x_0$  through  $\{E_k\}_k$* .

It is easy to see that if  $x_0$  is accessible through  $\{E_k\}_k$  and  $\{F_k\}_k \in [\{E_k\}_k]$ , then  $x_0$  is accessible through  $\{F_k\}_k$  as well. Furthermore,  $x_0 \in I[\{E_k\}_k]$ . Thus, we can extend the above definitions to ends. It was shown in [1] that if  $z_0 \in \partial \Omega$  is accessible, then it is accessible through some prime end  $[\{E_k\}_k]$  with  $I[\{E_k\}_k] = \{x_0\}$ . In addition, for all prime ends  $[\{E_k\}_k] \in \partial_{SP} \Omega$ , the point in  $I[\{E_k\}_k]$  is accessible through  $[\{E_k\}_k]$ .

However, as examples in [1] show, for some domains  $\Omega$ , not all points in  $\partial \Omega$  are accessible from  $\Omega$ , and it is not true that  $\partial_P \Omega$  is always compact. This has implications to the application of the Perron method in solving Dirichlet problems for the boundary  $\partial_P \Omega$ , and the goal of this paper is to find a way to overcome this lack of compactness; the key lemma in this direction is Lemma 4.6.



**Definition 2.13.** Let  $V \subset \Omega$  be an open connected set. We say that a point  $x_0 \in \partial\Omega$  is *accessible from the side of  $V$*  if there is a curve  $\gamma : [0, 1] \rightarrow \overline{\Omega}$  such that  $\gamma([0, 1)) \subset \Omega$ ,  $\gamma(1) = x_0$ , and for each positive integer  $n$  there is a real number  $t_n$  with  $1 - \frac{1}{n} < t_n < 1$  such that  $\gamma(t_n) \in V$ . We say that a chain  $\{E_k\}_k$  of  $\Omega$  is *from the side of  $V$*  if  $E_k \cap V$  is non-empty for each positive integer  $k$ .

Note that if  $\{E_k\}_k$  is from the side of  $V$ , and  $\{F_k\}_k \in [\{E_k\}_k]$ , then  $\{F_k\}_k$  is also from the side of  $V$ . Hence the property of being *from the side of  $V$*  is inherited from chains by ends.

**Remark 2.14.** In this paper, when we discuss curves  $\gamma$  that are locally rectifiable, we assume that  $\gamma$  is essentially arc-length parametrized; that is,  $\gamma : [0, \infty) \rightarrow X$  such that  $\gamma|_{[0, \ell(\gamma))}$  is arc-length parametrized, and if  $\ell(\gamma) < \infty$ , then for  $t \geq \ell(\gamma)$  we have  $\gamma(t) = \gamma(\ell(\gamma))$ . We call such parametrizations *standard parametrizations*.

Note that in Definitions 2.12 and 2.13, we could take  $\gamma$  to be maps from  $[0, \infty)$  rather than from  $[0, 1]$ . In this case, in Definition 2.13 we require  $\ell(\gamma) - 1/n < t_n < \ell(\gamma)$  whenever  $\ell(\gamma) < \infty$ , and  $n < t_n < \ell(\gamma)$  when  $\ell(\gamma) = \infty$ , rather than  $1 - 1/n < t_n < 1$ .

### 3. Preliminaries: Newton-Sobolev spaces and potential theory

We follow [22] in considering the Newtonian spaces as the analog of Sobolev spaces in the metric setting. Given a function  $u : X \rightarrow [-\infty, \infty]$ , we say that a non-negative Borel measurable function  $g$  on  $X$  is an *upper gradient* of  $u$  if whenever  $\gamma$  is a non-constant compact rectifiable curve in  $X$ , we have

$$|u(x) - u(y)| \leq \int_{\gamma} g \, ds,$$

where  $x$  and  $y$  denote the two end points of  $\gamma$ . The above inequality should be interpreted to mean that  $\int_{\gamma} g \, ds = \infty$  if at least one of  $u(x)$ ,  $u(y)$  is not finite. The notion of upper gradients is originally due to Heinonen and Koskela [14], where it was called a very weak gradient. Of course, if  $g$  is an upper gradient of  $u$  and  $\rho$  is a non-negative Borel measurable function on  $X$ , then  $g + \rho$  is also an upper gradient of  $u$ . If  $u$  has an upper gradient that belongs to  $L^p(X)$ , then the collection of all upper gradients of  $u$  in  $L^p(X)$  forms a convex subset of  $L^p(X)$ . Therefore, by the uniform convexity of  $L^p(X)$  when  $1 < p < \infty$  there is a unique function  $g_u \in L^p(X)$  that is in the  $L^p$ -closure of this convex set, with minimal norm. Such a function  $g_u$  is called the *minimal  $p$ -weak upper gradient* of  $u$ .

Given  $1 < p < \infty$ , the Newtonian space  $N^{1,p}(X)$  is the space

$$N^{1,p}(X) := \{u : X \rightarrow [-\infty, \infty] : \int_X |u|^p \, d\mu < \infty, \text{ has an upper gradient } g \in L^p(X)\} / \sim,$$

where the equivalence relationship  $\sim$  is such that  $u \sim v$  if and only if

$$\|u - v\|_{N^{1,p}(X)} := \left[ \int_X |u - v|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right]^{1/p} = 0,$$

the infimum being taken over all upper gradients  $g$  of  $u - v$ . See [22] or [3] for a discussion on the properties of  $N^{1,p}(X)$ . Just as sets of measure zero are exceptional sets in the  $L^p$ -theory, sets of  $p$ -capacity zero are exceptional sets in the potential theory associated with  $N^{1,p}(X)$ . Given a set  $A \subset X$ , its  *$p$ -capacity* is the number

$$C_p(A : X) := \inf_u \|u\|_{N^{1,p}(X)}^p,$$

where the infimum is taken over all  $u \in N^{1,p}(X)$  that satisfy  $u \geq 1$  on  $A$ .

**Definition 3.1.** We say that  $X$  supports a  $p$ -Poincaré inequality if there are constants  $C, \lambda \geq 1$  such that whenever  $u$  is a function on  $X$  with upper gradient  $g$  on  $X$  and  $B$  is a ball in  $X$ ,

$$\frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq C \operatorname{rad}(B) \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p d\mu \right)^{1/p}.$$

Here  $u_B$  denotes the integral average of  $u$  on  $B$ :

$$u_B := \frac{1}{\mu(B)} \int_B u d\mu.$$

Furthermore, we say that the measure  $\mu$  on  $X$  is doubling if there is a constant  $C \geq 1$  such that whenever  $B$  is a ball in  $X$ ,

$$\mu(2B) \leq C \mu(B).$$

**Assumption 3.2.** *Henceforth, in this paper we will assume that  $\mu$  is doubling and that  $X$  supports a  $p$ -Poincaré inequality. We refer the interested reader to [10] for an in-depth discussion on Poincaré inequalities. It was also shown in [10] that if  $X$  is complete,  $\mu$  is doubling, and  $X$  supports a  $p$ -Poincaré inequality, then  $X$  is quasiconvex. Given that the notions of prime ends, rectifiability of curves, and the metric topology are preserved under biLipschitz change in the metric, henceforth we will assume also that  $X$  is a geodesic space.*

**Definition 3.3.** Given a domain (open connected set)  $\Omega \subset X$ , the space of Newtonian functions with zero boundary values is the space

$$N_0^{1,p}(\Omega) := \{u \in N^{1,p}(X) : u = 0 \text{ in } X \setminus \Omega\}.$$

We refer the reader to [21] and the references therein for properties related to this function space. Given a function  $u$  defined only on  $\Omega$ , we say that  $u \in N_0^{1,p}(\Omega)$  if the zero-extension of  $u$  lies in  $N_0^{1,p}(\Omega)$ .

Finally, we introduce the concept of  $p$ -minimizers.

**Definition 3.4.** A function  $u \in N^{1,p}(\Omega)$  is said to be a  $p$ -minimizer in  $\Omega$  if it has minimal  $p$ -energy in  $\Omega$ . That is, for all  $\varphi \in N_0^{1,p}(\Omega)$ ,

$$\int_{\operatorname{supp}(\varphi)} g_u^p d\mu \leq \int_{\operatorname{supp}(\varphi)} g_{u+\varphi}^p d\mu.$$

Here,  $g_u$  and  $g_{u+\varphi}$  denote the minimal  $p$ -weak upper gradient of  $u$  and  $u + \varphi$  respectively. A function that satisfies this condition for nonnegative  $\varphi \in N_0^{1,p}(\Omega)$  is said to be a  $p$ -superminimizer in  $\Omega$ . A function is said to be  $p$ -harmonic in  $\Omega$  if it is a continuous  $p$ -minimizer in  $\Omega$ .

As the results in Kinnunen-Shanmugalingam [16] show, under the hypotheses considered in this paper, every  $p$ -minimizer can be modified on a set of  $p$ -capacity zero to obtain a locally Hölder continuous  $p$ -harmonic function.

The lower semicontinuous regularization of a function  $u$  is

$$u^*(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y).$$

As shown in [15], the equality  $u^* = u$  holds outside a set of zero  $p$ -capacity when  $u$  is a  $p$ -superminimizer. For this reason, any  $p$ -superminimizer used in this paper will be assumed to be lower semicontinuously regularized in this manner. Recall from the above Definition 3.3 that a function defined on  $\Omega$  is in  $N_0^{1,p}(\Omega)$  if its zero-extension to  $X \setminus \Omega$  is in  $N^{1,p}(X)$ .



**Definition 3.5.** Let  $V \subset X$  be open and bounded, with  $C_p(X \setminus V) > 0$ . Then, for  $f \in N^{1,p}(V)$  and  $\psi : V \rightarrow \overline{\mathbb{R}}$ , we define the set

$$\mathcal{K}_{\psi,f}(V) := \{v \in N^{1,p}(V) : v - f \in N_0^{1,p}(V), v \geq \psi \text{ a.e. in } V\}.$$

A function  $u \in \mathcal{K}_{\psi,f}(V)$  is said to be a *solution of the  $\mathcal{K}_{\psi,f}(V)$ -obstacle problem* if

$$\int_V g_u^p d\mu \leq \int_V g_v^p d\mu, \text{ for all } v \in \mathcal{K}_{\psi,f}(V).$$

It is shown in [15, Theorem 3.2] that solutions to the  $\mathcal{K}_{\psi,f}(V)$ -obstacle problem exist and are unique (in  $N^{1,p}(V)$ ), provided  $\mathcal{K}_{\psi,f}(V) \neq \emptyset$ .

Given a function  $f \in N^{1,p}(X)$  and a bounded domain  $\Omega \subset X$  with  $C_p(X \setminus \Omega) > 0$ , there is a unique function  $u \in N^{1,p}(X)$  such that  $u - f \in N_0^{1,p}(\Omega)$  and  $u$  is  $p$ -harmonic in  $\Omega$ . We denote this solution  $u$  by  $H_\Omega f$ . See [21] for the proofs of existence and uniqueness of such solutions. Note that  $H_\Omega f$  is the solution to the  $\mathcal{K}_{-\infty,f}(\Omega)$ -obstacle problem. The condition  $C_p(X \setminus \Omega) > 0$  is needed in order to have non-trivial solutions in  $\Omega$ . Should  $C_p(X \setminus \Omega) = 0$ , then  $N_0^{1,p}(\Omega) = N^{1,p}(X)$ , and in this case for every non-negative  $f \in N_0^{1,p}(\Omega)$  we would have that  $H_\Omega f$  be a non-negative  $p$ -harmonic function on  $X$  itself, and hence by the Harnack inequality (see [16]) we would have  $H_\Omega f = 0$ . By assuming  $C_p(X \setminus \Omega) > 0$  we avoid this problem.

Our setting in this paper will primarily be  $\overline{\Omega}^P$ . Since the subspace topology of  $\Omega$  inherited from  $\overline{\Omega}^P$  agrees with the standard metric topology on  $\Omega$  inherited from  $X$ , the Newton-Sobolev space  $N^{1,p}(\Omega)$  can be seen as the function space corresponding to both  $\Omega$ , seen as a domain in  $X$ , and  $\Omega$ , seen as a domain in  $\overline{\Omega}^P$ . The restriction of  $f \in N^{1,p}(X)$  to  $\Omega$  belongs to  $N^{1,p}(\Omega)$ , and thus the notation  $Hf := H_\Omega f$  is unambiguous.

However, one should keep in mind that in general functions in  $N^{1,p}(X)$ , when restricted to  $\Omega$ , may not have a natural extension to  $\partial_P \Omega$ . This is in contrast to the standard boundary  $\partial \Omega$ , where one can consider traces of Sobolev functions as discussed for example in [18] and [11]. In this paper we use  $Hf$  for such  $f$  only as an intermediate tool to study the Perron solutions adapted to  $\partial_P \Omega$ , but not as the end product itself.

## 4. Structure of the end and prime end boundaries

In this section we discuss some structures of the prime end boundary; these structures are useful in the subsequent sections where we consider the Perron method for the prime end boundary of a bounded domain. We first state two elementary lemmas regarding the geometry of chains. The proof of these lemmas use the properness of  $X$  (that is, closed and bounded subsets of  $X$  are compact).

**Lemma 4.1.** *Given a chain  $\{E_k\}_k$ , for every  $\varepsilon > 0$  there is an acceptable set  $E_j \in \{E_k\}_k$  such that*

$$E_j \subset N(I(\{E_k\}_k), \varepsilon).$$

**Lemma 4.2.** *Let  $\{E_k\}_k$  be a chain. Then, for every  $\varepsilon > 0$  and integer  $k$ , there is a connected component  $C_k^\varepsilon$  of  $N(I(\{E_k\}_k), \varepsilon) \cap E_k$  such that  $I(\{E_k\}_k) \subset \overline{C_k^\varepsilon}$ .*

The above lemmas can be proven by direct topological arguments and by using the definition of ends; we leave the proof to the interested reader. Next we prove two useful lemmas about the topology on  $\overline{\Omega}^P$ .

**Lemma 4.3.** *If  $\{x_k\}_k$  is a sequence of points in  $\Omega$  and  $[\{E_k\}_k] \in \partial_E \Omega$  such that  $x_k \rightarrow [\{E_k\}_k]$ , then no subsequence of  $\{x_k\}_k$  has a limit point in  $\Omega$ .*

*Proof.* Note that  $\bigcap_k \overline{E_k} \subset \partial \Omega$  and, for each positive integer  $j$ , the tail-end of the sequence  $\{x_k\}_k$  lies in  $E_j$ . Therefore, every cluster point of  $\{x_k\}_k$  must lie in  $\bigcap_k \overline{E_k} \subset \partial \Omega$ .  $\square$

**Lemma 4.4.** *If  $U \subset \overline{\Omega}^P$  is an open set in the prime end topology such that  $\partial_P \Omega \subset U$ , then for each  $[\{E_k\}_k] \in \partial_P \Omega$  and for each  $\{E_k\}_k \in [\{E_k\}_k]$ , there is a positive integer  $k_U$  such that  $E_{k_U} \subset U$ .*

*Proof.* We prove this lemma by contradiction. Suppose that  $\{E_k\} \in [\{E_k\}_k] \in \partial_P \Omega$  such that for each positive integer  $k$  we have  $E_k \not\subset U$ , that is, we can find  $x_k \in E_k \setminus U$ . It then follows that  $\{x_k\}_k$  is a sequence in  $\Omega$  with  $x_k \rightarrow [\{E_k\}_k]$ . But then, because  $U$  is open in the sequential topology of  $\Omega \cup \partial_P \Omega$  and  $[\{E_k\}_k] \in U$ , we must necessarily have a positive integer  $k_U$  such that whenever  $k \geq k_U$ ,  $x_k \in U$ , which contradicts the choice of  $x_k \in E_k \setminus U$ .  $\square$

Next, we prove a useful relation between  $\partial_{SP} \Omega$  and  $\partial_P \Omega$ .

**Theorem 4.5.** *With respect to the prime end topology on  $\overline{\Omega}^P$ ,  $\partial_{SP} \Omega$  is dense in  $\partial_P \Omega$ .*

*Proof.* As in Assumption 3.2, we assume that  $X$  is a geodesic space.

Given a prime end  $[\{E_k\}_k] \in \partial_P \Omega \setminus \partial_{SP} \Omega$ , fix a representative chain  $\{E_k\}_k$  of  $[\{E_k\}_k]$  such that  $E_n \subset N(I[\{E_k\}_k], \frac{1}{n})$ . Choose a sequence  $\{x_n\}_n$  in  $\Omega$  such that  $x_n \in E_n$  for each positive integer  $n$ .

For each  $x_n$ , let  $R_n = \text{dist}(x_n, X \setminus \Omega)$  and pick  $y_n \in \overline{B(x_n, R_n)} \cap \partial \Omega$ . Note that, since  $x_n \in N(I[\{E_k\}_k], \frac{1}{n})$ , we have that  $R_n \leq \frac{1}{n}$ .

Since  $X$  is a geodesic space and  $B(x_n, R_n) \subset \Omega$ , there is a geodesic  $\gamma_n : [0, R_n] \rightarrow \overline{\Omega}$  from  $x_n$  to  $y_n$  such that  $\gamma_n([0, R_n)) \subset B(x_n, R_n) \subset \Omega$ . Therefore,  $y_n$  is accessible and there is a prime end  $[\{F_k^n\}_k] \in \partial_{SP} \Omega$  such that  $I[\{F_k^n\}_k] = \{y_n\}$  and  $\gamma_n$  accesses  $y_n$  through  $[\{F_k^n\}_k]$  (see Definition 2.12). Though not relevant at the moment, for future use in the proof of Proposition 7.10 we note that, since  $B(x_n, R_n)$  is connected,  $d_M(x_n, [\{F_k^n\}_k]) = R_n \leq \frac{1}{n}$ . Furthermore, we can choose  $F_k^n$  so that  $\text{diam}(F_k^n) \leq 1/k$ .

We now prove that  $[\{F_k^n\}_k] \xrightarrow{\overline{\Omega}^P} [\{E_k\}_k]$ . Suppose this is not the case. Then there is a positive integer  $K$  such that, for each positive integer  $n$ , there is an integer  $j_n \geq n$  so that for each positive integer  $k$  we can find a point  $z_{j_n} \in F_k^{j_n} \setminus E_K$ . The choice of  $z_{j_n}$  does indeed depend on  $k$  as well, but since we next fix a choice of positive integer  $k$ , we do not indicate the dependance of  $z_{j_n}$  on  $k$  in the notation. Indeed, we now choose  $k \geq 2K + 2n$ .

On the other hand, for  $n \geq 2K$  we have  $x_{j_n} \in E_{K+1} \cap \gamma_{j_n}$ , and a set  $\beta_{j_n} = \gamma_{j_n} \cup F_k^{j_n}$ , containing  $x_{j_n}$  and  $z_{j_n}$ , with diameter  $\text{diam}(\beta_{j_n}) \leq 1/n + 1/k \leq 2/n$ . We now show that  $\beta_{j_n}$  is connected. We do not claim here that  $x_{j_n} \in F_k^{j_n}$ , but note that a point in  $\gamma_{j_n}$  lies in  $F_k^{j_n}$ , and a compact subcurve of  $\gamma_{j_n}$  therefore connects  $x_{j_n}$  to this point. Hence  $\beta_{j_n}$  is connected. Thus we have a point  $z_{j_n} \in \beta_{j_n}$  that lies outside  $E_K$ , and a point  $x_{j_n} \in E_{K+1} \cap \beta_{j_n}$ . It follows that  $\text{dist}_M(\Omega \cap \partial E_K, \Omega \cap \partial E_{K+1}) \leq 2/n$  for sufficiently large  $n$ . Letting  $n \rightarrow \infty$  we obtain that  $\text{dist}_M(\Omega \cap \partial E_K, \Omega \cap \partial E_{K+1}) = 0$ , which violates the definition of a chain. Hence we know that  $[\{F_k^n\}_k] \xrightarrow{\overline{\Omega}^P} [\{E_k\}_k]$ , completing the proof of the theorem.  $\square$

The next lemma provides a connection between locally rectifiable curves of infinite length and ends that are, in some sense, from the side of those curves.

**Lemma 4.6.** *Let  $\Omega$  be a bounded domain in  $X$ . Suppose that  $\gamma$  is a curve in  $\Omega$  such that*

$$I(\gamma) := \bigcap_{n \in \mathbb{N}} \overline{\gamma((n, \infty))} \subset \partial\Omega,$$

*and set*

$$E(\gamma) := \{[\{F_k\}_k] \in \partial_E \Omega : \forall k \in \mathbb{N}, \exists t_k \text{ such that } \gamma([t_k, \infty)) \subset F_k\}.$$

*As in Assumption 4.7 below we consider the order relation  $\leq$  on  $E(\gamma)$  defined by  $x \leq y$  if and only if  $x|y$ . Then  $E(\gamma)$  has a minimal (or, least) element  $[\{E_k\}_k]$ . Furthermore, for each  $[\{F_k\}_k] \in E(\gamma)$  we have  $[\{E_k\}_k]$  divides  $[\{F_k\}_k]$  and  $I[\{E_k\}_k] = I(\gamma)$ .*

*Proof.* For each positive integer  $k$  let  $E_k$  denote the connected component of the set  $N_M(\gamma((k, \infty)), 1/k) \cap \Omega$  that contains the tail-end  $\gamma((k, \infty))$  of  $\gamma$ . We will show that the end corresponding to the chain  $\{E_k\}_k$  should be a minimal end in  $E(\gamma)$ .

It is easily seen that  $[\{E_k\}_k] \in E(\gamma)$ . So it suffices to show that whenever  $[\{F_k\}_k] \in E(\gamma)$ , the end  $[\{E_k\}_k]$  divides  $[\{F_k\}_k]$ . To do so, let  $[\{F_k\}_k] \in E(\gamma)$ . We want to show that given a positive integer  $k$  there is a positive integer  $j_k$  such that  $E_{j_k} \subset F_k$ .

Suppose that the above is not true. Then for each positive integer  $j$  the set  $E_j \setminus F_k$  is non-empty. By the construction of  $E_j$ , for any  $x \in E_j \setminus F_k$  there is a real number  $t_x \in [j, \infty)$  with  $d_M(x, \gamma(t_x)) < 1/j$ . Note that by the definition of chains,  $\text{dist}_M(\partial F_k \cap \Omega, \partial F_{k+1} \cap \Omega) > 0$ . So we can choose a positive integer  $J$  such that  $1/J < \text{dist}_M(\partial F_k \cap \Omega, \partial F_{k+1} \cap \Omega)$ . Consider  $j \geq J$ , and fix  $x_j \in E_j \setminus F_k$ , and set  $t_j := t_{x_j}$ . We then have

$$d_M(x_j, \gamma(t_j)) < 1/j \leq 1/J < \text{dist}_M(\partial F_k \cap \Omega, \partial F_{k+1} \cap \Omega),$$

It follows now from the fact that  $x_j \notin F_k$  that  $\gamma(t_j) \notin F_{k+1}$ . Consequently, for each positive integer  $j > J$  we can find a real number  $t_j \geq j$  such that  $\gamma(t_j) \notin F_{k+1}$ . Thus no tail end of  $\gamma$  can lie in  $F_{k+1}$ , which violates the fact that  $[\{F_k\}_k] \in E(\gamma)$ .

Hence we can conclude that necessarily there is some positive integer  $j_k$  such that  $E_{j_k} \subset F_k$ , that is, the end  $[\{E_k\}_k]$  divides  $[\{F_k\}_k]$ , concluding the proof.  $\square$

In addition to our previous assumptions on  $X$ , we also will assume for the remainder of the paper that the domain  $\Omega$  fulfills the following property:

**Assumption 4.7.** *For every collection  $\mathcal{F}$  of ends that is totally ordered by division such that  $x \leq y$  if and only if  $x|y$ , there is an end  $[\{G_k\}_k]$  such that  $[\{G_k\}_k] \leq [\{F_k\}_k]$  for every  $[\{F_k\}_k] \in \mathcal{F}$ .*

The above assumption essentially states that we assume that the collection of all ends of  $\Omega$  satisfies the hypotheses of Zorn's lemma.

Should  $\Omega$  be a simply connected bounded planar domain, the above condition is seen to hold true. The proof of this fact there goes through the Riemann mapping theorem; in more general settings it is not clear to us whether the above condition automatically holds. However, in many situations this condition is directly verifiable. If  $\partial_{SP} \Omega$  is compact, then by Theorem 4.5 we know that  $\partial_P \Omega = \partial_{SP} \Omega$ , and in this case the fact that above assumption holds is a consequence of the results found in [1, Section 7]. Indeed, by the results in [1], it follows that given an end  $[\{E_k\}_k]$ , every point in  $I[\{E_k\}_k]$  is accessible through  $[\{E_k\}_k]$  by rectifiable curves, and hence a prime end from  $\partial_{SP} \Omega$  divides  $[\{E_k\}_k]$ .

Under the assumption of 4.7, we have the following fact about  $\overline{\Omega}^E$ .

**Theorem 4.8.** *Suppose that  $\Omega$  is a bounded domain satisfying the assumption given in Definition 4.7. Let  $[\{E_k\}_k]$  be an end of  $\Omega$ . Then there is a prime end  $[\{F_k\}_k]$  of  $\Omega$  that divides  $[\{E_k\}_k]$ .*

*Proof.* Consider the set  $\mathcal{E}$  of ends that divide  $[\{E_k\}_k]$ , ordered by division. If this set contains only  $[\{E_k\}_k]$ , then  $[\{E_k\}_k]$  is a prime end.

Assume  $\mathcal{E}$  has more than one element. Let  $\mathcal{F}$  be a totally ordered subset of  $\mathcal{E}$ , indexed by a corresponding totally ordered set  $A$ . By the assumption given in the theorem, there is an element  $[\{G_k\}_k]$  that divides all the elements of  $\mathcal{F}$ . Since each of these elements divides  $[\{E_k\}_k]$  in turn,  $[\{G_k\}_k] \mid [\{E_k\}_k]$ . Thus,  $[\{G_k\}_k] \in \mathcal{E}$ , satisfying the conditions for the use of Zorn's Lemma. Thus  $\mathcal{E}$  has a minimal element, and this minimal element is necessarily a prime end.  $\square$

Finally, we prove the following consequence of the assumptions made on  $\Omega$  earlier in this section. This will be integral to our results in the next section.

**Lemma 4.9.** *Let  $\Omega$  be a bounded domain satisfying the assumption given in Definition 4.7, and let  $\gamma$  be a curve in  $\Omega$  such that  $\bigcap_{n \in \mathbb{N}} \overline{\gamma((n, \infty))} \subset \partial\Omega$ . Then there is a prime end  $[\{A_k\}_k]$  such that  $[\{A_k\}_k] \mid [\{F_k\}_k]$  for every  $[\{F_k\}_k] \in E(\gamma)$ , and  $A_k \cap \gamma \neq \emptyset$  for every integer  $k$ .*

One cannot in general expect this prime end to be in  $E(\gamma)$ , as the harmonic comb example shows. Thus the best possible link the prime end has to  $\gamma$  is the condition that  $A_k \cap \gamma \neq \emptyset$  for every integer  $k$ .

*Proof.* By Lemma 4.6, we know that  $E(\gamma)$  has a minimal element  $[\{G_k\}_k]$  and that  $[\{G_k\}_k] \mid [\{F_k\}_k]$  for every  $[\{F_k\}_k] \in E(\gamma)$ .

If  $[\{G_k\}_k]$  happens to be a prime end, then set  $[\{A_k\}_k] = [\{G_k\}_k]$  and the proof would be complete. If not, we may use Theorem 4.8 to obtain a prime end  $[\{H_k\}_k]$  that divides  $[\{G_k\}_k]$ . Since  $[\{G_k\}_k]$  is minimal in  $E(\gamma)$ , it must be the case that  $[\{H_k\}_k] \notin E(\gamma)$ . Now we have one of two following possibilities:

- (a) For every  $k \in \mathbb{N}$ , there is a positive real number  $t_k$  such that  $\gamma(t_k) \in H_k$ .
- (b) There exists a positive integer  $k_0$  such that  $H_{k_0} \cap \gamma = \emptyset$ .

If  $[\{H_k\}_k]$  behaves as in (a), we simply take  $[\{A_k\}_k] = [\{H_k\}_k]$  and the proof is complete. We now show that possibility (b) does not occur. We may also, without loss of generality, suppose that  $\overline{H_k} \cap \Omega = H_k$ .

Assume that  $[\{H_k\}_k]$  behaves as in (b). For simplicity, we may take  $k_0 = 1$ . Then, define

$$m_H := \text{dist}_M(\partial H_1 \cap \Omega, \partial H_2 \cap \Omega)$$

and

$$\hat{H}_k := \left( \bigcup_{x \in H_2} B_M(x, (1 - \frac{1}{k+1})m_H) \right) \cap H_1.$$

Then  $H_2 \subset \hat{H}_k \subset \hat{H}_{k+1} \subset H_1$  and

$$\text{dist}_M(\partial \hat{H}_k \cap \Omega, \partial \hat{H}_{k+1} \cap \Omega) > 0$$

for every  $k \in \mathbb{N}$ . Because  $\gamma$  does not intersect  $H_1$  and  $H_1$  is relatively closed in  $\Omega$  by assumption, we have that  $\gamma \subset \Omega \setminus \overline{H_1}$ .

Finally, we define  $D_k$  as the connected component of  $G_k \setminus \overline{\hat{H}_k}$  that contains the tail end of  $\gamma$  inside  $G_k$ . Since  $\overline{\hat{H}_k}$  contains no points of  $\gamma$ , we know that this component exists. By construction,  $D_k \supset D_{k+1}$  and  $\bigcap_{k \in \mathbb{N}} \overline{D_k} \subset \partial\Omega$ . As before, we

need only show that  $\text{dist}_M(\partial D_k \cap \Omega, \partial D_{k+1} \cap \Omega) > 0$  for all  $k \in \mathbb{N}$  to establish that  $\{D_k\}_k$  is a chain. Let

$$M_k = \min\{\text{dist}_M(\partial \widehat{H}_k \cap \Omega, \partial \widehat{H}_{k+1} \cap \Omega), \text{dist}_M(\partial G_k \cap \Omega, \partial G_{k+1} \cap \Omega)\}.$$

Note that  $M_k > 0$ . Take  $x \in \partial D_k \cap \Omega$  and  $y \in \partial D_{k+1} \cap \Omega$  and consider the following cases.

**Case 1:**  $x \in \partial G_k \cap \Omega$  and  $y \in \partial G_{k+1} \cap \Omega$ . In this case, we immediately have that  $d_M(x, y) \geq M_k$ .

**Case 2:**  $x \in \partial G_k \cap \Omega$  and  $y \in \partial \widehat{H}_{k+1} \cap \Omega$ , but  $y \notin \partial G_{k+1} \cap \Omega$ . Here, it must be the case that  $y \in G_{k+1}$ . So  $d_M(x, y) \geq M_k$ .

**Case 3:**  $x \in \partial \widehat{H}_k \cap \Omega$  and  $y \in \partial \widehat{H}_{k+1} \cap \Omega$ . As in Case 1, we immediately have that  $d_M(x, y) \geq M_k$ .

**Case 4:**  $x \in \partial \widehat{H}_k \cap \Omega$  and  $y \in \partial G_{k+1} \cap \Omega$ , but  $y \notin \partial \widehat{H}_{k+1} \cap \Omega$ . Here, it must be that  $y \in G_k \setminus \widehat{H}_{k+1}$ . So  $d_M(x, y) \geq M_k$ .

**Case 5:**  $x \notin \Omega \cap (\partial G_k \cup \partial \widehat{H}_k)$  or  $y \notin \Omega \cap (\partial G_{k+1} \cup \partial \widehat{H}_{k+1})$ . We will focus on the first possibility, the second being handled in a very similar manner. Since  $x \notin \Omega \cap (\partial G_k \cup \partial \widehat{H}_k)$ , it follows that  $x$  is in the interior of  $G_k$ , and hence in the interior of  $G_k \setminus \widehat{H}_k$ . It follows that for sufficiently small  $r > 0$  the connected set  $B_M(x, r) \subset G_k \setminus \widehat{H}_k$ , which then means that  $B_M(x, r) \subset D_k$ , violating the fact that  $x \in \partial D_k$ . Hence this case is not possible.

The above argument allows us to conclude that  $\text{dist}_M(\partial D_k \cap \Omega, \partial D_{k+1} \cap \Omega) > 0$ , and so  $\{D_k\}_k$  is a chain and  $[\{D_k\}_k]$  is an end. By construction,  $[\{D_k\}_k] \perp [\{G_k\}_k]$  and  $[\{D_k\}_k] \in E(\gamma)$ . Because  $[\{G_k\}_k]$  divides each end in  $E(\gamma)$ , and so  $[\{D_k\}_k] = [\{G_k\}_k]$ . However, by construction  $[\{H_k\}_k]$  does not divide  $[\{D_k\}_k]$ , which violates the choice of  $[\{H_k\}_k]$  as a prime end that divides  $[\{G_k\}_k]$ . Hence the alternative (b) cannot occur. This completes the proof of the lemma.  $\square$

## 5. Prime ends are from all sides.

The goal of this section is to show that if  $V \subset \Omega$  is an open connected set such that  $\partial V \cap \partial \Omega$  is non-empty, then there is a prime end from the side of  $V$ . To do so we employ the inner metric  $d_{\text{inn}}^V$  (see Definition 2.1).

For each  $\varepsilon > 0$  let  $V_\varepsilon := \{x \in V : \text{dist}(x, X \setminus V) > \varepsilon\}$ , and for a (locally rectifiable) curve  $\gamma$  of infinite length in  $X$ , let

$$I(\gamma) := \bigcap_{n \in \mathbb{N}} \overline{\gamma((n, \infty))}.$$

Note that if  $\gamma \subset \overline{V}$ , then  $I(\gamma)$  is a connected compact subset of  $\overline{V}$ .

**Lemma 5.1.** *Let  $V \subset \Omega$  be an open connected set and suppose that  $x_\infty \in \partial V \cap \partial \Omega$ . Let  $\{x_k\}_k$  be a sequence of points in  $V$  such that  $\lim_k x_k = x_\infty$ , and let  $x_0 \in V$ . Suppose that for each positive integer  $k$  the  $d_{\text{inn}}^V$ -geodesic  $\gamma_{x_0, x_k}^V$  does not intersect  $\partial \Omega$ . Then there is a curve  $\gamma : [0, \infty) \rightarrow \overline{V}$  such that  $\gamma$  is the local uniform limit of a subsequence of the sequence of curves  $\{\gamma_{x_0, x_k}^V\}$ . Furthermore, if  $\gamma$  has infinite length, then  $I(\gamma) \subset \partial V$ .*

**Remark 5.2.** Note that if there are two points  $z, w \in V$  such that the  $d_{\text{inn}}^V$ -geodesic connecting  $z$  to  $w$  intersects  $\partial \Omega$ , then, because this geodesic has finite length (with respect to the metric  $d$ ), it follows that there is a point  $x_0 \in \partial V \cap \partial \Omega$  that is accessible from the side of  $V$ . See Definition 2.13 for the definition of “accessibility from the side” of  $V$ . As a consequence, if such points  $z, w$  exist, then there is a

prime end from the side of  $V$ , and we can choose this prime end from the class  $\partial_{SP}\Omega$ .

*Proof of Lemma 5.1.* The existence of the curve  $\gamma$  is easily given by applying the Arzelà-Ascoli theorem to the equibounded (since  $\Omega$  is bounded) equicontinuous (since these curves are 1-Lipschitz maps with respect to the underlying metric  $d$ ) family  $\{\gamma_{x_0, x_k}^V\}$ . It is also clear that  $I(\gamma) \subset \overline{V}$  and that each subcurve of  $\gamma$  is a  $d_{\text{inn}}^V$ -geodesic between its endpoints. Demonstrating that when  $\gamma$  has infinite length  $I(\gamma) \subset \partial V$  requires slightly more work.

We argue by contradiction. Suppose that there is a point  $y \in I(\gamma) \cap V$ , and pick a sequence  $\{y_i\}$  with  $y_i \rightarrow y$  and  $y_i \in \gamma([i, \infty))$  for each  $i$ . Since  $V$  is open, there is a sufficiently small neighborhood of  $y$  within  $V$  such that the metrics  $d$  and  $d_{\text{inn}}^V$  are biLipschitz equivalent inside this neighborhood. Thus,  $y_i$  converges to  $y$  with respect to  $d_{\text{inn}}^V$ , requiring that  $d_{\text{inn}}^V(y_i, y)$  be uniformly bounded by some  $M < \infty$ .

Recall that  $d_{\text{inn}}^V(x_0, y)$  must be finite; denote this quantity by  $N$ . By the triangle inequality,

$$d_{\text{inn}}^V(x_0, y_i) \leq d_{\text{inn}}^V(x_0, y) + d_{\text{inn}}^V(y_i, y) \leq N + M.$$

Since  $M$  and  $N$  are independent of  $i$ , we have that  $d_{\text{inn}}^V(x_0, y_i)$  is uniformly bounded. But we picked  $y_i$  such that  $y_i \in \gamma([i, \infty))$ , and since  $\gamma$  is locally a geodesic with infinite length,  $d_{\text{inn}}^V(x_0, y_i) \geq i$  for each  $i$ . But this contradicts the above bound on  $d_{\text{inn}}^V(x_0, y_i)$ . Thus, we have that  $y \notin V$ , that is,  $I(\gamma) \subset \partial V$ .  $\square$

**Theorem 5.3.** *Let  $\Omega$  be a bounded connected open set satisfying the condition given in Assumption 4.7, and  $V \subset \Omega$  be an open, connected set. If  $\partial\Omega \cap \partial V$  is non-empty, then there is a prime end of  $\Omega$  from the side of  $V$ .*

*Proof.* If there is a rectifiable curve in  $\overline{V}$  that connects a point in  $V$  to a point in  $\partial V \cap \partial\Omega$ , then the accessibility results of [1] gives a corresponding prime end from the side of  $V$ . See also Remark 5.2 above. Hence, without loss of generality, we may assume that there is no rectifiable curve in  $\overline{V}$  that connects some point in  $V$  to a point in  $\partial\Omega \cap \partial V$ . Note that we now fulfill the assumptions of Lemma 5.1, allowing its use in the remainder of the proof.

We fix  $x_0 \in V$  and  $x_\infty \in \partial\Omega \cap \partial V$ , and let  $\{x_k\}_k$  be a sequence of points in  $V$  such that  $\lim_k x_k = x_\infty$ . For each positive integer  $k$  let  $\gamma_{x_0, x_k}^V$  be as in the statement of Lemma 5.1. Clearly  $\gamma_{x_0, x_k}^V$  cannot intersect  $\partial\Omega$  because if it does, then we have a point in  $\partial V \cap \partial\Omega$  that is accessible from  $V$ , violating the assumption stated in the previous paragraph of this proof; see Remark 5.2. Hence Lemma 5.1 gives us a locally rectifiable curve  $\gamma : [0, \infty) \rightarrow \overline{V}$  such that  $\gamma(0) = x_0$ , and for each  $t > 0$  the curve  $\gamma|_{[0, t]}$  is a  $d_{\text{inn}}^V$ -geodesic that lies inside  $\Omega$ . Since we assumed that there are no rectifiable curves connecting a point in  $V$  to  $\partial V \cap \partial\Omega$ ,  $\gamma$  must have infinite length. Thus,  $I(\gamma) \subset \partial V$ .

Now the proof diverges according to two possibilities.

**Case 1:**  $I(\gamma) \subset \partial\Omega \cap \partial V$ . Then we can proceed to construct an end as follows. For  $k \in \mathbb{N}$  we set  $E_k$  to be the connected component of  $N(I(\gamma), 1/k) \cap \Omega$  that contains  $\gamma([t_k, \infty))$  for some  $t_k > 0$ . Each  $E_k$  is an acceptable set, and  $\{E_k\}_k$  satisfies the conditions of a chain. Note that

$$\text{dist}_M(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) \geq \text{dist}(\Omega \cap \partial E_k, \Omega \cap \partial E_{k+1}) \geq \frac{1}{k(k+1)} > 0$$

and that  $I[\{E_k\}_k] = I(\gamma) \subset \partial\Omega \cap \partial V \subset \partial\Omega$ .

It is clear that  $[\{E_k\}_k]$  is from the side of  $V$ , however there is no *a priori* reason for  $[\{E_k\}_k]$  to be prime. Note, however, that  $[\{E_k\}_k]$  is clearly a member of  $E(\gamma)$ , and so by Lemma 4.9 there is a prime end  $[\{A_k\}_k]$  dividing  $[\{E_k\}_k]$  such that  $A_k \cap \gamma \neq \emptyset$  for each  $k$ . Therefore, for each  $k$ ,  $A_k \cap \overline{V}$  must be nonempty.



Furthermore, we can choose each  $A_k$  to be open (see [1, Remark 4.5]), and so we conclude that  $A_k \cap V$  is nonempty. It follows that  $\{A_k\}_k$  is a prime end from the side of  $V$ .

**Case 2:**  $I(\gamma) \not\subset \partial\Omega \cap \partial V$ . We denote by  $\overline{B}(x, r)$  the closed ball  $\{y \in X : d(x, y) \leq r\}$  rather than the closure of the open ball  $B(x, r)$ . Recall from Assumption 3.2 that  $X$  is a geodesic space. It follows that whenever  $x \in X$  and  $r > 0$ , each pair of points  $z, w \in \overline{B}(x, r)$  can be connected in the closed ball  $\overline{B}(x, r)$  by a curve of length at most  $2r$ . Again, because  $X$  is doubling and hence separable, we can cover  $\partial V \setminus \partial\Omega$  by at most a countable family of balls  $B(z_i, r_i)$  with  $z_i \in \partial V \setminus \partial\Omega$  and  $r_i = \min\{\text{dist}(z_i, X \setminus \Omega), d(z_i, x_0)\}/10$ . Setting

$$V_j := V \cup \bigcup_{i=1}^j B(z_i, r_i)$$

for positive integers  $j$ , note that if  $x, y \in V$ , then

$$d(x, y) \leq d_{\text{inn}}^{V_{j+1}}(x, y) \leq d_{\text{inn}}^{V_j}(x, y) \leq d_{\text{inn}}^V(x, y). \quad (5.1)$$

As in the first part of the proof, we obtain curves  $\gamma_j$  for each  $j$  that are locally uniform limits of  $d_{\text{inn}}^{V_j}$ -geodesics connecting  $x_0$  to  $x_k$ . Because of (5.1), and because each  $\gamma_n$  is a  $d_{\text{inn}}^{V_n}$ -geodesic, we know that if  $\gamma_m(t_j) \in B(z_j, r_j) \cap V$  for some  $m \leq j$ , then for all  $n \geq j$  we have that  $\gamma_n([t_j + 2r_j, \infty))$  does not intersect  $\overline{B}(z_j, r_j)$ . It follows that for  $n \geq j$  we have that  $I(\gamma_n) \cap \overline{B}(z_j, r_j)$  is empty. Therefore,

$$I(\gamma_n) \subset \partial V \setminus \bigcup_{i=1}^n B(z_i, r_i).$$

A final application of Arzelà-Ascoli theorem gives a subsequence of  $\{\gamma_n\}_n$  that converges locally uniformly to a curve  $\beta : [0, \infty) \rightarrow \bigcup_j \overline{V_j}$  such that  $\beta(0) = x_0$ , and because for each  $n \in \mathbb{N}$  we have that  $\beta([t_n + 2r_n, \infty)) \cap \overline{B}(z_n, r_n)$  is empty,

$$I(\beta) \subset \partial V \setminus \bigcup_{i \in \mathbb{N}} B(z_i, r_i) = \partial V \cap \partial\Omega.$$

The proof is now completed by applying the argument at the end of the proof of Case 1 to  $\beta$  instead of  $\gamma$ .  $\square$

The following corollary to the above theorem gives us a useful fact, namely that compact containment of connected sets in  $\Omega$  is the same in both the Prime End topology and the topology on  $\Omega$  inherited from  $X$ . Note however that if we do not require  $V$  to be connected, the following theorem would be false in general.

**Corollary 5.4.** *Let  $V \subset \Omega$  be an open, connected set. Then  $\overline{V} \subset \Omega$  if and only if  $\overline{V}^{P, \Omega} \subset \Omega$ .*

*Proof.* If  $\overline{V} \subset \Omega$ , then  $\overline{V} = \overline{V}^{P, \Omega}$ . If  $\overline{V}^{P, \Omega} \subset \Omega$ , then clearly there can be no prime ends from the side of  $V$ . Thus, by Theorem 5.3,  $\partial V \cap \partial\Omega = \emptyset$ . Therefore,  $\overline{V} \subset \Omega$ .  $\square$

## 6. Prime End Capacity and Newtonian Spaces

Recall that in Section 3 a notion of  $p$ -capacity associated with the space  $N^{1,p}(X)$  was discussed. In this section we will modify this notion to take into consideration the structure of  $\Omega \subset \overline{\Omega}^P$ . This new version of  $p$ -capacity is useful in the study of the Perron method adapted to the prime end boundary of  $\Omega$ .

**Definition 6.1.** For  $E \subset \overline{\Omega}^P$  let

$$\overline{C}_p^P(E) = \inf_{u \in \mathcal{A}_E} \|u\|_{N^{1,p}(\Omega)}^p,$$

where  $u \in \mathcal{A}_E$  if  $u \in N^{1,p}(\Omega)$  satisfies both  $u \geq 1$  on  $E \cap \Omega$  and

$$\liminf_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} u(y) \geq 1 \text{ for all } x \in E \cap \partial_P \Omega.$$

In the above definition, we can impose the additional requirement that  $0 \leq u \leq 1$  without any change in the resulting number for  $E$ .

The capacity  $\overline{C}_p^P$  satisfies the usual basic properties of a capacity.

**Lemma 6.2.** Let  $E, E_1, E_2, E_3, \dots$  be arbitrary subsets of  $\overline{\Omega}$ . Then

- (a)  $\overline{C}_p^P(\emptyset) = 0$ ,
- (b)  $\mu(E \cap \Omega) \leq \overline{C}_p^P(E)$ ,
- (c) If  $E_1 \subset E_2$ , then  $\overline{C}_p^P(E_1) \leq \overline{C}_p^P(E_2)$  (monotonicity),
- (d)  $\overline{C}_p^P\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \overline{C}_p^P(E_i)$  (countable subadditivity).

The proof of the above lemma follows precisely the proof of an analogous result in [6, Proposition 3.2], and so is omitted here.

We say that a function  $u$  on  $X$  is  $p$ -quasicontinuous (or, quasicontinuous) on an open set  $W \subset X$  if for every  $\varepsilon > 0$  we can find an open set  $U_\varepsilon \subset X$  such that  $u|_{W \setminus U_\varepsilon}$  is continuous and  $C_p(U_\varepsilon) < \varepsilon$ .

**Proposition 6.3.** Suppose that the measure on  $X$  is doubling and supports a  $p$ -Poincaré inequality. Then  $\overline{C}_p^P$  is an outer capacity, i.e. for all  $E \subset \overline{\Omega}^P$ ,

$$\overline{C}_p^P(E) = \inf_G \overline{C}_p^P(G),$$

where the infimum is taken over all  $G \supset E$  that are open in  $\overline{\Omega}^P$ .

While the proof of this proposition is very similar to the proof of the related result [6, Proposition 3.3], the situation considered by [6] was simpler in that the boundary of the domain considered there was the so-called Mazurkiewicz boundary, and so the function  $w$  defined there in a manner analogous to the proof here is easily seen to be admissible in computing the capacity. Here additional arguments were needed, and so for the convenience of the reader we provide the complete proof here.

*Proof.* By the assumptions on  $X$  (doubling property of  $\mu$  and the support of a  $p$ -Poincaré inequality) and by the results in [22] and [5], we know that functions in  $N^{1,p}(X)$  and functions in  $N^{1,p}(\Omega)$  are  $p$ -quasicontinuous.

By the monotonicity of  $\overline{C}_p^P$ , we obtain the inequality  $\overline{C}_p^P(E) \leq \inf_G \overline{C}_p^P(G)$  for free. We must work harder for the reverse inequality.

Given  $E \subset \overline{\Omega}^P$  and  $\varepsilon > 0$ , we pick a function  $u \in \mathcal{A}_E$  with  $0 \leq u \leq 1$  such that

$$\|u\|_{N^{1,p}(\Omega)} \leq \overline{C}_p^P(E)^{1/p} + \varepsilon.$$

Since  $u$  is quasicontinuous on  $\Omega$ , we may also take some open set  $V \subset \Omega$  such that  $C_p(V)^{1/p} \leq \varepsilon$  and  $u|_{\Omega \setminus V}$  is continuous. Thus,  $\{x \in \Omega : u(x) > 1 - \varepsilon\} \setminus V$  is an open set in  $\Omega \setminus V$  with respect to the subspace topology. Therefore there is another open set  $U \subset \Omega$  such that

$$U \setminus V = \{x \in \Omega : u(x) > 1 - \varepsilon\} \setminus V \supset (E \cap \Omega) \setminus V.$$

Because  $C_p(V) \leq \varepsilon^p$ , we can choose  $v \in N^{1,p}(X)$  satisfying  $\|v\|_{N^{1,p}(X)} < 2\varepsilon$ ,  $0 \leq v \leq 1$  on  $X$ , and  $v \geq 1$  on  $V$ . Set

$$w = \frac{u}{1 - \varepsilon} + v.$$

Then  $w \geq 1$  on  $U \cup V$ , which is an open set containing  $E \cap \Omega$ . Also, for each  $[\{E_k\}_k] \in E \cap \partial_P \Omega$ , there is a positive integer  $K$  such that  $u > 1 - \varepsilon$  on  $E_K$ . Indeed, if not, then we can find a sequence of points  $x_k \in E_k$  such that  $u(x_k) \leq 1 - \varepsilon$  but  $x_k \xrightarrow{\overline{\Omega}^P} [\{E_k\}_k] \in E \cap \partial_P \Omega$ , a violation of the choice of  $u \in \mathcal{A}_E$ .

Let

$$W = U \cup V \cup \bigcup_{[\{E_k\}_k] \in E \cap \partial_P \Omega} (E_K \cup E_K^P),$$

where  $E_K^P$  is as defined in Remark 2.11. Then  $W \supset E$  is an open set in  $\overline{\Omega}^P$  and  $w \in \mathcal{A}_W$ . So

$$\begin{aligned} \overline{C}_p^P(E)^{1/p} &\leq \inf_G \overline{C}_p^P(G)^{1/p} \leq \overline{C}_p^P(W)^{1/p} \leq \|w\|_{N^{1,p}(\Omega^P)} \\ &\leq \frac{1}{1 - \varepsilon} \|u\|_{N^{1,p}(\Omega^P)} + \|v\|_{N^{1,p}(\Omega^P)} \leq \frac{1}{1 - \varepsilon} (\overline{C}_p^P(E)^{1/p} + \varepsilon) + 2\varepsilon. \end{aligned}$$

By letting  $\varepsilon \rightarrow 0$ , the proof is complete.  $\square$

We also restate the definition of quasicontinuity with respect to this capacity.

**Definition 6.4.** A function  $f : \overline{\Omega}^P \rightarrow \mathbb{R}$  is  $\overline{C}_p^P$ -quasicontinuous if, for every  $\varepsilon > 0$ , there is a relatively open set  $U \subset \overline{\Omega}^P$  such that  $\overline{C}_p^P(U) < \varepsilon$  and  $f|_{\overline{\Omega}^P \setminus U}$  is real-valued continuous.

It is natural for us to try to further relate  $\overline{C}_p^P$  to the usual capacity  $C_p$ . To do this in a meaningful way, we would require a method of relating subsets of  $\overline{\Omega}^P$  to those in  $\overline{\Omega}$ . Since single elements in  $\overline{\Omega}^P$  might correspond to large sets in  $\overline{\Omega}$ , there is no easy mapping between  $\overline{\Omega}^P$  and  $\overline{\Omega}$  as in the case of the Mazurkiewicz boundary in [6]. Instead, we introduce the notion of the *Prime End Pushforward* of a set  $E \subset \overline{\Omega}$  in the following way.

**Definition 6.5.** Given  $E \subset X$ , the  $\Omega$ -Prime End Pushforward of  $E$ , denoted  $P(E)$ , is defined as

$$P(E) := (E \cap \Omega) \cup \{[\{E_k\}] \in \partial_P \Omega \mid I[\{E_k\}] \subset E\}.$$

It is clear from the definition that if  $E \subset F$ , then  $P(E) \subset P(F)$ . Also, this pushforward can easily be shown to be “an open map”, in that if  $E$  is open in  $X$ , then  $P(E)$  is open in  $\overline{\Omega}^P$ . Hence, if  $E \subset \overline{\Omega}$  is relatively open, then  $P(E)$  is open in  $\overline{\Omega}^P$ .

With this definition, we have the following Lemma. Recall that we assume the measure on  $X$  to be doubling and support a  $p$ -Poincaré inequality.

**Lemma 6.6.** Let  $E \subset X$ . Then

$$\overline{C}_p^P(P(E)) \leq C_p(E).$$

*Proof.* Given any  $\varepsilon > 0$ , we may pick an open set  $G \supset E$  in  $X$  such that  $C_p(G) \leq C_p(E) + \varepsilon/2$ . This is due to the fact that  $C_p$  is an outer capacity (see [5, Corollary 1.3] or [3, Theorem 5.21]).

Let  $f \in N^{1,p}(X)$  such that  $f = 1$  on  $G$  and  $\|f\|_{N^{1,p}(X)}^p < C_p(G) + \varepsilon/2$ . Define  $\tilde{f} := f|_\Omega$ . Note that  $\tilde{f} \in N^{1,p}(\Omega)$ .

Immediately, if  $x \in P(G) \cap \Omega = G \cap \Omega$ , then  $\tilde{f}(x) = 1$ . For  $x \in P(G) \cap \partial_P \Omega$ , we must look at a sequence  $\{y_k\}_k$  in  $\Omega$  converging to  $x$  in  $\overline{\Omega}^P$ . Since  $P(G)$  is open, we may assume that  $y_k \in P(G) \cap \Omega$  for each  $k$ . Then, clearly,  $\liminf_{y_k \xrightarrow{\overline{\Omega}^P} x} \tilde{f}(y_k) \geq 1$ . Thus,

$\tilde{f}$  is an admissible function for the computation of  $\overline{C}_p^P(P(G))$ . So,

$$\overline{C}_p^P(P(E)) \leq \overline{C}_p^P(P(G)) \leq \|\tilde{f}\|_{N^{1,p}(\Omega)}^p \leq C_p(E) + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$ , the proof is completed.  $\square$

Finally, in order to compare boundary values of functions on  $\overline{\Omega}^P$ , we need to consider  $N_0^{1,p}(\Omega)$  as given in Definition 3.3.

The following proposition is analogous to Proposition 5.4 of [6]. The major difference between our situation here and that of [6] is that there is no continuous map  $\Phi : \partial_P \Omega \rightarrow \partial \Omega$ , and so the proof of the following proposition is more complicated than that found in [6].

**Proposition 6.7.** *If  $f \in N_0^{1,p}(\Omega)$ , then the zero-extension of  $f$  to  $\partial_P \Omega$  is  $\overline{C}_p^P$ -quasicontinuous.*

*Proof.* Let  $f^0$  be the zero extension of  $f$  (as a function on  $\Omega$ ) to all of  $X$ . Then  $f^0 \in N^{1,p}(X)$ , and so for any  $\varepsilon > 0$  there is an open set  $U_\varepsilon$  in  $X$  such that  $C_p(U_\varepsilon) < \varepsilon$  and  $f^0|_{X \setminus U_\varepsilon}$  is continuous. Now let  $\hat{f} : \overline{\Omega}^P \rightarrow \mathbb{R}$  be defined as

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \partial_P \Omega. \end{cases}$$

By Lemma 6.6 we know that  $\overline{C}_p^P(P(U_\varepsilon)) < \varepsilon$ . We wish to show that  $\hat{f}|_{\overline{\Omega}^P \setminus P(U_\varepsilon)}$  is continuous. Let  $x \in \overline{\Omega}^P \setminus P(U_\varepsilon)$  and  $\{y_k\}_k$  be a sequence in  $\overline{\Omega}^P \setminus P(U_\varepsilon)$  such that  $y_k \xrightarrow{\overline{\Omega}^P} x$ . We wish to check that  $\hat{f}(y_k) \rightarrow \hat{f}(x)$ . Since  $\hat{f}|_{\Omega \setminus P(U_\varepsilon)} = f|_{\Omega \setminus P(U_\varepsilon)}$  is continuous, we know that if  $x \in \Omega$  then the above convergence holds. So without loss of generality, we may consider the following two cases.

**Case 1:**  $y_k \in \partial_P \Omega$  for each  $k$ , and  $x \in \partial_P \Omega$ . Since  $\hat{f}(y_k) = 0 = \hat{f}(x)$  for all  $k$ , this case is immediate.

**Case 2:**  $y_k \in \Omega$  for each  $k$ , and  $x \in \partial_P \Omega$ . Let  $\{y_{k_i}\}_i$  be a subsequence of  $\{y_k\}_k$ . Since  $I[x]$  is a compact set and  $\overline{\Omega}$  is a compact subset of  $X$ , there is a further subsequence  $\{y_{k_{i,j}}\}_j$  such that, for some  $x_0 \in I[x]$ ,  $y_{k_{i,j}} \rightarrow x_0$  in the topology of  $X$ . Since  $y_{k_{i,j}} \in X \setminus U_\varepsilon$  for each  $j$  and  $X \setminus U_\varepsilon$  is a closed set, the limit  $x_0$  of  $y_{k_{i,j}}$  cannot lie in  $U_\varepsilon$ . Therefore  $x_0 \in X \setminus (\Omega \cup U_\varepsilon)$ ,  $f^0(x_0) = 0$  and  $\hat{f}(y_{k_{i,j}}) = f^0(y_{k_{i,j}}) \rightarrow 0$ . Since this happens for all subsequences of  $\{y_k\}_k$ , we conclude that  $\hat{f}(y_{k_{i,j}}) \rightarrow 0 = \hat{f}(x)$ .

With both possibilities dealt with, we have proved the desired claim.  $\square$

It is possible to define a dual notion to the *Prime End Pushforward* of set, namely the *Prime End Pullback*.

**Definition 6.8.** Given  $F \subset \overline{\Omega}^P$ , the  $\Omega$ -*Prime End Pullback* of  $F$ , denoted  $P^{-1}(F)$ , is defined as

$$P^{-1}(F) := (E \cap \Omega) \cup \bigcup_{\{E_k\} \in F} I[\{E_k\}].$$

It is natural to consider how the two notions interact. It can be quickly deduced from their definitions that if  $E \subset X$  and  $F \subset \overline{\Omega}^P$ , then  $P^{-1}(P(E)) \subset E$  and  $F \subset P(P^{-1}(F))$ . As the following two examples show, equality does not hold in general for either case.

**Example 6.9.** If we take  $X = \mathbb{R}^2$ , with

$$\Omega := (0, 1)^2 \setminus \bigcup_{n=2}^{\infty} \{1/n\} \times (0, 1/2],$$

and let  $E = [0, 1]^2$ , we observe that

$$P^{-1}(P(E)) = [0, 1]^2 \setminus \{0\} \times [0, 1/2).$$

Thus,  $E \not\subset P^{-1}(P(E))$  in this case.

**Example 6.10.** Letting  $X = \mathbb{R}^2$  and let  $\Omega$  be the slit disk

$$\Omega = B(0, 1) \setminus [0, 1) \times \{0\}.$$

Take (recalling Remark 2.11)

$$F = \{(x, y) \in \Omega \mid y > 0\}^P.$$

Then  $F \subset \overline{\Omega}^P$  consists of the upper half of the slit disk in addition to the prime ends associated with the 'top' part of the slit. It is then easy to see that  $P(P^{-1}(F))$  will contain both 'sides' of the slit, and so  $P(P^{-1}(F)) \not\subset F$ .

The proof of the following lemma is mutatis mutandis the same as the proof of Lemma 6.6. We leave it to the interested reader to verify this.

**Lemma 6.11.** *Given  $E \subset \overline{\Omega}^P$ , we have*

$$\overline{C}_p^P(E) \leq C_p(P^{-1}(E)).$$

## 7. The Perron solution with respect to Prime Ends

Now we are ready to consider the following Dirichlet problem: Given  $g : \partial_P \Omega \rightarrow \mathbb{R}$ , find a function  $u$  that is  $p$ -harmonic on  $\Omega$  and such that  $u = g$  on  $\partial_P \Omega$  in some sense. The method we use to construct possible solutions to the above problem for certain type of functions  $g$  is the Perron method, adapted to  $\partial_P \Omega$ . We continue to assume the standard assumptions about  $X$  (the doubling property of the measure on  $X$ , and the validity of a  $p$ -Poincaré inequality on  $X$ ), and that  $\Omega$  is a bounded domain in  $X$  with  $C_p(X \setminus \Omega) > 0$  such that  $\Omega$  satisfies the condition given in Definition 4.7.

**Definition 7.1.** A function  $u : \Omega \rightarrow (-\infty, \infty]$  is said to be  $p$ -superharmonic if

- (a)  $u$  is lower semicontinuous,
- (b)  $u$  is not identically  $\infty$  on  $\Omega$ ,
- (c) for every nonempty open set  $V \Subset \Omega$  and all functions  $v \in Lip(X)$ , if  $v \leq u$  on  $\partial V$ , then  $H_V v \leq u$  in  $V$ .

A function  $u$  is said to be  $p$ -subharmonic if  $-u$  is  $p$ -superharmonic.

We now define the Perron solution with respect to  $\overline{\Omega}^P$ .

**Definition 7.2.** Given a function  $f : \partial_P \Omega \rightarrow \overline{\mathbb{R}}$ , let  $\mathcal{U}_f(\overline{\Omega}^P)$  be the set of all  $p$ -superharmonic functions  $u$  on  $\Omega$  bounded below such that

$$\liminf_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} [\{E_n\}_n]} u(y) \geq f([\{E_n\}_n]) \text{ for all } [\{E_n\}_n] \in \partial_P \Omega.$$

We define the *upper Perron solution* of  $f$  by

$$\overline{P}_{\overline{\Omega}^P} f(x) = \inf_{u \in \mathcal{U}_f(\overline{\Omega}^P)} u(x), \quad x \in \Omega.$$

Similarly, let  $\mathcal{L}_f(\overline{\Omega}^P)$  be the set of all  $p$ -subharmonic functions  $u$  on  $\Omega$  bounded above such that

$$\limsup_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} [\{E_n\}_n]} u(y) \leq f([\{E_n\}_n]) \text{ for all } [\{E_n\}_n] \in \partial_P \Omega.$$

We define the *lower Perron solution* of  $f$  by

$$\underline{P}_{\overline{\Omega}^P} f(x) = \sup_{u \in \mathcal{L}_f(\overline{\Omega}^P)} u(x), \quad x \in \Omega^P.$$

Note that  $\underline{P}_{\overline{\Omega}^P} f = -\overline{P}_{\overline{\Omega}^P}(-f)$ . If  $\overline{P}_{\overline{\Omega}^P} f = \underline{P}_{\overline{\Omega}^P} f$  on  $\Omega$ , then we let  $P_{\overline{\Omega}^P} f := \overline{P}_{\overline{\Omega}^P} f$ , and  $f$  is said to be *resolutive*.

For the classical formulation of the Perron solution, it is shown in [4, Theorem 5.1] that functions  $f \in N^{1,p}(X)$  are resolutive. We wish to provide a similar result for an appropriate class of functions on  $\partial_P \Omega$ . Due to the potential non-compactness of the space  $\overline{\Omega}^P$ , we must first prove that several important results still hold in this space. Chief among them is the following comparison principle. An analogous comparison principle, set up for the Mazurkiewicz boundary in [6, Proposition 7.2], is straightforward to prove because of the assumption in [6] that the Mazurkiewicz boundary  $\partial_M \Omega$  is compact. Here we overcome the lack of compactness of  $\partial_P \Omega$  with the aid of Corollary 5.4.

**Proposition 7.3.** *Assume that  $u$  is  $p$ -superharmonic and that  $v$  is  $p$ -subharmonic in  $\Omega$ . If*

$$\infty \neq \limsup_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} v(y) \leq \liminf_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} u(y) \neq -\infty \text{ for each } x \in \partial_P \Omega,$$

*then  $v \leq u$  in  $\Omega$ .*

*Proof.* Let  $\varepsilon > 0$ . Since  $u$  is lower semicontinuous and  $v$  is upper semicontinuous, we know that  $V_\varepsilon := \{y \in \Omega : u(y) - v(y) > -\varepsilon\}$  is an open subset of  $\Omega$ .

By the assumption of this proposition, for each  $x \in \partial_P \Omega$  we can find a neighborhood  $V_\varepsilon^x$  of  $x$  in  $\overline{\Omega}^P$  such that  $v < u + \varepsilon$  in  $V_\varepsilon^x \cap \Omega$ . Note that  $V_\varepsilon^x \cap \Omega \subset V_\varepsilon$  for each  $x \in \partial_P \Omega$ . Thus  $V_\varepsilon \cup \partial_P \Omega$  is an open subset of  $\overline{\Omega}^P$ .

Let  $U_\varepsilon = \overline{\Omega}^P \setminus \overline{V_\varepsilon}$  and  $C_\varepsilon$  be a connected component of  $U_\varepsilon$ . Then, by Lemma 4.4,  $\overline{C_\varepsilon}^{P,\Omega} \subset \Omega$  and  $v \leq u + \varepsilon$  on  $\partial_P^\Omega C_\varepsilon$ .

By Corollary 5.4, we know that  $\overline{C_\varepsilon} \subset \Omega$ , and, since  $\partial_P^\Omega C_\varepsilon = \partial C_\varepsilon$  in this case,  $v \leq u + \varepsilon$  on  $\partial C_\varepsilon$ . We now proceed as in [15, Theorem 7.2] to see that  $v \leq u$  in  $C_\varepsilon$ . Since this inequality holds for each connected component of  $U_\varepsilon$ , we conclude that  $v \leq u + \varepsilon$  in  $U_\varepsilon$ . Letting  $\varepsilon \rightarrow 0$ , the proof is complete.  $\square$

An immediate consequence of Proposition 7.3 is the following Corollary.



**Corollary 7.4.** *If  $f : \partial_P \Omega \rightarrow \mathbb{R}$ , then*

$$\underline{P}_{\overline{\Omega}^P} f \leq \overline{P}_{\overline{\Omega}^P} f.$$

Before we state our main theorem, we first need the following two results. The first part of Proposition 7.6 is proved in [23], while the second part is proved in [4, Proposition 5.5]. See [9] for more on convergence properties related to obstacle problems. The proof of Lemma 7.5 is very similar to the proof of the analogous result [6, Lemma 7.5], and so we omit the proof here.

**Lemma 7.5.** *Let  $\{U_k\}_{k=1}^\infty$  be a decreasing sequence of relatively open sets in  $\overline{\Omega}^P$  such that  $\overline{C}_p^P(U_k) < 2^{-kp}$ . Then there exists a decreasing sequence of nonnegative functions  $\{\psi_j\}_{j=1}^\infty$  on  $\Omega$  such that  $\|\psi_j\|_{N^{1,p}(\Omega)} < 2^{-j}$ ,  $\psi_j \geq k - j$  in  $U_k \cap \Omega$ , and  $\lim_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} \psi_j(y) \geq k - j$  for all  $x \in U_k \cap \partial_P \Omega$ .*

**Proposition 7.6.** *Let  $\{f_j\}_{j=1}^\infty$  be a  $p$ -quasieverywhere decreasing sequence of functions in  $N^{1,p}(\Omega)$  such that  $f_j \rightarrow f$  in  $N^{1,p}(\Omega)$  as  $j \rightarrow \infty$ . Then  $Hf_j$  decreases to  $Hf$  locally uniformly in  $\Omega$ .*

*If  $u$  and  $u_j$  are solutions to the  $\mathcal{K}_{f,f}$  and  $\mathcal{K}_{f_j,f_j}$ -obstacle problems, then  $\{u_j\}_{j=1}^\infty$  decreases q.e. in  $\Omega$  to  $u$ .*

We now state the main theorem of this paper.

**Theorem 7.7.** *Let  $f : \overline{\Omega}^P \rightarrow \overline{\mathbb{R}}$  be a  $\overline{C}_p^P$ -quasicontinuous function such that  $f|_\Omega$  is in  $N^{1,p}(\Omega)$ . Then  $f$  is resolutive and  $P_{\overline{\Omega}^P} f = Hf$ .*

Having overcome the drawback from the fact that  $\partial_P \Omega$  may not be compact with the help of Proposition 7.3, the proof of the above main theorem is very similar to that of [6, Theorem 7.4]. However, one difference still remains—namely the topology of  $\overline{\Omega}^P$  near the boundary  $\partial_P \Omega$ , which is not as simple as that of the Mazurkiewicz boundary. Hence we provide most of the details of the proof of Theorem 7.7 here.

*Proof.* First, we assume that  $f \geq 0$ . We extend  $Hf$  to  $\overline{\Omega}^P$  by letting  $Hf = f$  on  $\partial_P \Omega$ . We now show that this extension is  $\overline{C}_p^P$ -quasicontinuous.

Let  $h = f - Hf$ . Then  $h \in N_0^{1,p}(\Omega)$  is quasicontinuous on  $\Omega$  with  $\overline{C}_p^P$ -quasicontinuous extension  $h = 0$  to  $\partial_P \Omega$ , see Proposition 6.7. Because  $f$  is  $\overline{C}_p^P$ -quasicontinuous on  $\overline{\Omega}^P$ , it now follows that so is  $Hf$ .

Pick open sets  $\{G_j\}$  in  $\overline{\Omega}^P$  with  $\overline{C}_p^P(G_j) < 2^{-jp}$  such that  $Hf|_{\overline{\Omega}^P \setminus G_j}$  is continuous. Defining  $U_k = \bigcup_{j=k+1}^\infty G_j$ , we see that  $\overline{C}_p^P(U_k) < 2^{-kp}$  and  $Hf|_{\overline{\Omega}^P \setminus U_k}$  is still continuous.

These sets  $\{U_k\}$  fulfill the conditions of Lemma 7.5, and so we may take functions  $\{\psi_j\}$  as described in that Lemma. We set  $f_j = Hf + \psi_j$  (note here that  $f_j$  is a function on  $\Omega$  alone) and let  $\varphi_j$  be the lower semicontinuously regularized solution of the  $\mathcal{K}_{f_j,f_j}(\Omega)$ -obstacle problem as given in Definition 3.5.

For each positive integer  $m$  we have that

$$f_j \geq \psi_j \geq m \text{ on } U_{m+j} \cap \Omega.$$

Given  $\varepsilon > 0$ , let  $x \in \partial_P \Omega$ . If  $x \notin U_{m+j}$ , by the continuity of  $Hf|_{\overline{\Omega}^P \setminus U_{m+j}}$ , there is a neighborhood  $V_x$  of  $x$  in  $\overline{\Omega}^P$  such that

$$f_j(y) \geq Hf(y) \geq Hf(x) - \varepsilon = f(x) - \varepsilon \text{ for all } y \in (V_x \cap \Omega) \setminus U_{m+j}.$$

So, if  $x \in \partial_P \Omega \setminus U_{m+j}$ ,

$$f_j \geq \min\{f(x) - \varepsilon, m\} \text{ in } V_x \cap \Omega^P.$$

If, instead,  $x \in U_{m+j}$ , we take  $V_x = U_{m+j}$ .

Now, as in the proof of [6, Theorem 7.4] we have that  $\varphi_j(y) \geq \min\{f(x) - \varepsilon, m\}$  for all  $y \in V_x \cap \Omega$ . Therefore,

$$\liminf_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} \varphi_j(y) \geq \min\{f(x) - \varepsilon, m\}.$$

As  $\varepsilon \rightarrow 0$  and  $m \rightarrow \infty$ , we have that

$$\liminf_{\Omega \ni y \xrightarrow{\overline{\Omega}^P} x} \varphi_j(y) \geq f(x) \text{ for all } x \in \partial_P \Omega.$$

Since  $\varphi_j$  is  $p$ -superharmonic, we have that  $\varphi_j \in \mathcal{U}_f(\overline{\Omega}^P)$ , and so  $\varphi_j \geq \overline{P}_{\overline{\Omega}^P} f$ . Because  $Hf$  is the solution to the  $\mathcal{K}_{Hf, Hf}(\Omega)$ -obstacle problem, by Proposition 7.6 we know that  $\varphi_j$  decreases quasieverywhere to  $Hf$ , that is,  $\overline{P}_{\overline{\Omega}^P} f \leq Hf$  q.e. in  $\Omega$  when  $f \geq 0$ .

Note that if  $f \in N^{1,p}(\Omega)$  has a  $\overline{C}_p^P$ -quasicontinuous extension to  $\overline{\Omega}^P$ , then so does  $\max\{f, m\}$  for each integer  $m$ . Therefore, for  $f \in N^{1,p}(\Omega)$ , not necessarily non-negative,

$$\overline{P}_{\overline{\Omega}^P} f \leq \lim_{m \rightarrow -\infty} \overline{P}_{\overline{\Omega}^P} \max\{f, m\} \leq \lim_{m \rightarrow \infty} H \max\{f, m\} = Hf \text{ q.e. in } \Omega.$$

Because  $\overline{P}_{\overline{\Omega}^P} f$  is  $p$ -harmonic in  $\Omega$  (the proof of this fact, for the Dirichlet problem corresponding to the metric boundary, can be found in [4, Section 4]; the proof there goes through in our setting verbatim since the modifications considered there in the proof occur only on relatively compact subsets of  $\Omega$  itself) and hence is continuous, we have that both  $\overline{P}_{\overline{\Omega}^P} f$  and  $Hf$  are continuous. Therefore  $\overline{P}_{\overline{\Omega}^P} f \leq Hf$  everywhere in  $\Omega$ .

Finally, with the aid of Proposition 7.3, or more precisely, with the help of Corollary 7.4, we see that

$$\underline{P}_{\overline{\Omega}^P} f = -\overline{P}_{\overline{\Omega}^P}(-f) \geq -H(-f) = Hf \geq \overline{P}_{\overline{\Omega}^P} f \geq \underline{P}_{\overline{\Omega}^P} f.$$

Thus  $Hf = \underline{P}_{\overline{\Omega}^P} f = \overline{P}_{\overline{\Omega}^P} f$  and  $f$  is resolutive.  $\square$

The following results show that solution  $P_{\overline{\Omega}^P} f$  is stable under perturbation of  $f$  on a set of  $\overline{C}_p^P$  capacity zero. Some of these results have analogous results for the Mazurkiewicz boundary in [6], but the conclusions found there are stronger in general, due to the fact that the hypothesis that the Mazurkiewicz boundary (which is the same as  $\partial_{SP} \Omega$ ) is compact is much stronger than the assumptions in our paper. For example, in the setting of [6] *all* continuous functions are resolutive, whereas in our setting these continuous functions are in addition required to be Lipschitz on  $\partial_{SP} \Omega$ . The results here are not duplicates of those in [6] because the domains considered here do not in general have  $\partial_{SP} \Omega$  compact, and so the results found in our paper are valid for a larger class of domains than the results of [6], including the setting of [6].

**Proposition 7.8.** *Let  $f : \overline{\Omega}^P \rightarrow \overline{\mathbb{R}}$  be a  $\overline{C}_p^P$ -quasicontinuous function with  $f|_{\Omega}$  in the class  $N^{1,p}(\Omega)$ . If  $h : \partial_P \Omega \rightarrow \overline{\mathbb{R}}$  is zero  $\overline{C}_p^P$  quasi-everywhere, then  $f + h$  is resolutive with respect to  $\overline{\Omega}^P$ , and  $P_{\overline{\Omega}^P}(f + h) = P_{\overline{\Omega}^P}(f)$ .*

*Proof.* We may extend  $h$  into  $\Omega$  by zero, and clearly  $h|_{\Omega} \in N^{1,p}(\Omega)$ . Note that, since  $\bar{C}_p^P$  is an outer capacity (see Lemma 6.3), this extended function  $h$  is  $\bar{C}_p^P$ -quasicontinuous. Thus  $f + h$  is  $\bar{C}_p^P$ -quasicontinuous. Finally,  $(f + h)|_{\Omega} \in N^{1,p}(\Omega)$ , so by using Theorem 7.7,  $f + h$  is resolutive and  $P_{\bar{\Omega}^P}(f + h) = H(f + h)$ . Since  $f = f + h$  in  $\Omega$ , we therefore have  $Hf = H(f + h)$ . Thus, by Theorem 7.7 again,

$$P_{\bar{\Omega}^P}(f + h) = H(f + h) = Hf = P_{\bar{\Omega}^P}f.$$

□

**Corollary 7.9.** *Let  $f : \bar{\Omega}^P \rightarrow \bar{\mathbb{R}}$  be a bounded  $\bar{C}_p^P$ -quasicontinuous function with  $f|_{\Omega} \in N^{1,p}(\Omega)$  and  $u$  be a bounded  $p$ -harmonic function in  $\Omega$ . If  $E \subset \partial_P \Omega$  such that  $\bar{C}_p^P(E) = 0$  and, for all  $x \in \partial_P \Omega \setminus E$ ,*

$$\lim_{\Omega \ni y \xrightarrow{\bar{\Omega}^P} x} u(y) = f(x),$$

*then  $u = P_{\bar{\Omega}^P}f$ .*

*Proof.* Since both  $f$  and  $u$  are bounded, we may (simultaneously) rescale them such that  $0 \leq f, u \leq 1$ . Then we know that  $u \in \mathcal{U}_{f-\chi_E}(\bar{\Omega}^P)$  and  $u \in \mathcal{L}_{f+\chi_E}(\bar{\Omega}^P)$ . Thus, by the preceding proposition,

$$u \leq \underline{P}_{\bar{\Omega}^P}(f + \chi_E) = P_{\bar{\Omega}^P}f = \bar{P}_{\bar{\Omega}^P}(f - \chi_E) \leq u.$$

□

Finally, as an application of the above resolvitivity results, we discuss issues of resolvitivity of continuous functions on  $\partial_P \Omega$ . Note that by the results in [4], continuous functions on  $\partial \Omega$  are resolutive. However, in the setting of  $\partial_P \Omega$  we are unable to get such a general result. However, we are able to get resolvitivity for certain types of continuous functions on  $\partial_P \Omega$ . This is the focus of the the remaining part of this paper.

Recall that  $\partial_{SP} \Omega$  denotes the collection of all prime ends whose impression contains only one point. As discussed in Section 2 (see also [1]), this set is equipped with a metric  $d_M$ , the extension of the Mazurkiewicz metric on  $\Omega$ .

**Proposition 7.10.** *Let  $f : \partial_P \Omega \rightarrow \mathbb{R}$  be continuous on  $\partial_P \Omega$  and  $d_M$ -Lipschitz continuous on  $\partial_{SP} \Omega$ . Then  $f$  is resolutive. Furthermore, if  $h : \partial_P \Omega \rightarrow \bar{\mathbb{R}}$  is zero  $\bar{C}_p^P$  quasi-everywhere, then  $f + h$  is resolutive with respect to  $\bar{\Omega}^P$ , and  $P_{\bar{\Omega}^P}(f + h) = P_{\bar{\Omega}^P}(f)$ .*

*Proof.* By an application of the McShane extension theorem (see [12]), we extend  $f$  to a function  $F : \bar{\Omega}^P \rightarrow \mathbb{R}$  such that  $F = f$  on  $\partial_P \Omega$  and  $F$  is  $d_M$ -Lipschitz on  $\Omega \cup \partial_{SP} \Omega$ .

We now show that  $F$  is continuous on  $\bar{\Omega}^P$ . By construction,  $F|_{\Omega \cup \partial_{SP} \Omega}$  is continuous. Since  $F = f$  on  $\partial_P \Omega$ , we also see that  $F|_{\partial_P \Omega}$  is also continuous. It remains to show that given any end  $[\{E_k\}_k] \in \partial_P \Omega \setminus \partial_{SP} \Omega$  and a sequence  $\{x_n\}_n$  in  $\Omega$  with  $x_n \xrightarrow{\bar{\Omega}^P} [\{E_k\}_k]$ , we have  $F(x_n) \rightarrow F([\{E_k\}_k])$ .

At first, we will prove our result only for sequences  $x_n \xrightarrow{\bar{\Omega}^P} [\{E_k\}_k]$  such that, for each  $n$ ,  $x_n \in N(I[\{E_k\}_k], \frac{1}{n})$ . In addition, we will fix a representative chain  $\{E_k\}_k \in [\{E_k\}_k]$  such that, for all  $n \geq k$ ,  $x_n \in E_k$ . Recall also that we assume  $X$  to be a geodesic space.

By modifying the proof of Theorem 4.5, we obtain a sequence  $\{\{F_k^n\}_k\}_n$  in  $\partial_{SP}\Omega$  such that  $\{F_k^n\}_k \xrightarrow{\overline{\Omega}^P} \{E_k\}_k$  and  $d_M(x_n, \{F_k^n\}_k) \leq \frac{1}{n}$ .

Since  $F$  is continuous on  $\partial_P\Omega$ , we know that  $F(\{F_k^n\}_k) \rightarrow F(\{E_k\}_k)$ . Given any  $\varepsilon$ , we may pick a large-enough positive integer  $N$  such that

$$|F(\{F_k^N\}_k) - F(\{E_k\}_k)| < \frac{\varepsilon}{2}$$

and

$$|F(x_N) - F(\{F_k^N\}_k)| \leq L d_M(x_N, \{F_k^N\}_k) \leq \frac{L}{N} \leq \frac{\varepsilon}{2},$$

where  $L$  is the  $d_M$ -Lipschitz constant for  $F$  on  $\Omega \cup \partial_{SP}\Omega$ . Then

$$|F(x_N) - F(\{E_k\}_k)| \leq |F(\{F_k^N\}_k) - F(\{E_k\}_k)| + |F(x_N) - F(\{F_k^N\}_k)| \leq \varepsilon.$$

Thus,  $F(x_n) \rightarrow F(\{E_k\}_k)$ .

Now, given any arbitrary sequence  $\{x_n\}$  of points in  $\Omega$  such that  $x_n \xrightarrow{\overline{\Omega}^P} \{E_k\}_k$ , consider  $\{|F(x_n) - F(\{E_k\}_k)|\}_n$ . Given any subsequence of  $\{x_n\}$ , we may pick a further subsequence  $\{z_n\}$  such that  $z_n \in N(I[\{E_k\}_k], \frac{1}{n})$ . Therefore,  $|F(z_n) - F(\{E_k\}_k)| \rightarrow 0$ , implying that  $|F(x_n) - F(\{E_k\}_k)| \rightarrow 0$ , which completes the proof of continuity of  $F$ .

Now an application of the main theorem above yields the resolvitivity of  $F$ , and hence the resolvitivity of  $f$ , completing the proof of the first part of the proposition.

The second part now follows from an application of Proposition 7.8 to the function  $F$ .  $\square$

**Remark 7.11.** Observe that in the above proposition, we can relax the condition of  $f$  being continuous on  $\partial_P\Omega$  to  $f$  being  $\overline{C}_p^P$ -quasicontinuous on  $\partial_P\Omega$ , the remaining (Lipschitz) condition of  $f$  also holding. More precisely, if for each  $\varepsilon > 0$  we can find an open set  $U_\varepsilon \subset \overline{\Omega}^P$  with  $\overline{C}_p^P(U_\varepsilon) < \varepsilon$  such that  $f|_{[\partial_P\Omega \setminus (U_\varepsilon)] \cup \partial_{SP}\Omega}$  is continuous and  $f$  is  $d_M$ -Lipschitz continuous on  $\partial_{SP}\Omega$ , then  $f$  is resolvitive.

## 8. Some examples

The use of prime ends in the Perron method also yields new results about Euclidean domains. For example, in the classical Dirichlet problem where the boundary considered is the topological (that is, metric) boundary of the domain, it is well-known that if  $f : \partial\Omega \rightarrow \mathbb{R}$  is continuous and  $E \subset \partial\Omega$  such that  $C_p(E) = 0$ , then any bounded perturbation of  $f$  on  $E$  would yield a resolutive function whose Perron solution agrees with the Perron solution of  $f$  in  $\Omega$ ; see [13] for a proof of this in the weighted Euclidean setting, and for more general metric measure spaces as considered in this paper, see [4] for this fact. The prime end boundary approach studied in this paper gives a larger set  $E$  on which the value of the boundary data would be irrelevant in the above sense of perturbation.

The goal of this section is to give three such example domains in Euclidean setting.

**Example 8.1.** The first example we discuss in this section is that of the harmonic comb, also known as the topologist's comb. This example was extensively studied in [2]. This comb is a simply connected planar domain given by

$$\Omega := (0, 1) \times (0, 1) \setminus \bigcup_{n \in \mathbb{N}} \{1/n\} \times [0, 1/2].$$

It was shown in [2] that given a function on  $\partial\Omega$ , continuous and bounded on  $\partial\Omega \setminus \{0\} \times [0, 1/2)$ , any perturbation of the function on the set  $E := \{0\} \times [0, 1/2)$  yields a resolutive function whose Perron solution coincides with the Perron solution of the original function. Note that  $C_p(E) > 0$  for  $p > 1$ , but  $\bar{C}_p^P(P(E)) = 0$ . Note also that the prime end boundary in this case is the same as the singleton prime end boundary  $\partial_{SP}\Omega$ . Hence the “prime end-Perron solution” of any boundary data defined on  $\partial\Omega$  is independent of the values of the boundary data on  $E$  as long as the boundary data is Lipschitz (with respect to the Mazurkiewicz metric  $d_M$ ) continuous on the part of the boundary of  $\Omega$  that arises as impressions of prime ends. On the other hand, if  $f$  is a quasicontinuous function on  $\bar{\Omega} \setminus \{0\} \times [0, 1/2)$  (not necessarily bounded) such that  $f|_{\Omega} \in N^{1,p}(\Omega)$ , then  $f|_{\partial\Omega \setminus \{0\} \times [0, 1/2]}$  is resolutive, and any perturbation of  $f$  on a set  $F \subset \partial\Omega$  with  $\bar{C}_p^P(P(F)) = 0$  yields the same Perron solution. Hence the results obtained from the perspective of prime end boundaries are complementary to the results in [2].

In the above example none of the points in  $E$  belongs to the impression of any prime end. However,  $\bar{C}_p^P(P(E))$  does make sense. We point out here that the results of [2] are related to another type of Perron solution, namely, the Perron solution with respect to the topological boundary  $\partial\Omega$ . The difference between the two types of Perron solutions in this instance is that in the case of the prime end boundary, the condition on the superharmonic functions is not enforced at any of the points in  $E$ . This is more in line with the behavior of functions in  $N^{1,p}$ -classes; boundary points that are not accessible via rectifiable curves from within the domain ought not to influence the Dirichlet problem for the domain.

As a consequence of the results of the previous section (see Remark 7.11), if we know that  $\bar{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$ , then any bounded function on  $\partial_P\Omega$  that is Lipschitz continuous on  $\partial_{SP}\Omega$  with respect to the Mazurkiewicz metric  $d_M$  is resolutive, and any bounded perturbation of such a function on  $\partial_P\Omega \setminus \partial_{SP}\Omega$  yields a resolutive function whose Perron solution agrees with the Perron solution of the original function. This phenomenon is illustrated by the following two examples.

**Example 8.2.** This example considers the so-called double comb:

$$\Omega := (0, 1) \times (0, 1) \setminus \bigcup_{1 < n \in \mathbb{N}} \{1/(2n)\} \times [0, 1 - 1/n] \cup \{1/(2n + 1)\} \times [1/n, 1].$$

This again is a simply connected planar domain, but now the set  $E := \{0\} \times [0, 1]$  is the impression of a single prime end. Note that  $\partial_P\Omega$  is compact in this example, but  $\partial_{SP}\Omega$  is not. It is again easy to see (by the use of functions  $u_\varepsilon := \varepsilon d_{\text{inn}}^\Omega(x_0, \cdot)$  for each  $\varepsilon > 0$  and for a fixed  $x_0 \in \Omega$ ) that  $\bar{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$ , although  $C_p(P^{-1}(\partial_P\Omega \setminus \partial_{SP}\Omega)) > 0$ . It follows that any function on  $\partial_P\Omega$  that is Lipschitz continuous on  $\partial_{SP}\Omega$  (with respect to  $d_M$ ) is resolutive, and any perturbation of this function on  $E$  is also resolutive. Strictly speaking, each individual point in  $E$  does not form a separate prime end boundary; the entire set  $E$  is the impression of a prime end. Therefore, in the above statement, by “perturbation on  $E$ ” we mean perturbing the value of the function by changing its value to a different one on the entire set  $E$ . However, we can relax this “constant on  $E$ ” requirement in the following sense. Any function on the topological boundary  $\partial\Omega$  that is Lipschitz continuous on  $\partial\Omega \setminus E$  is resolutive, and perturbations of such functions on  $E$  would yield the same Perron solution. A similar statement holds for functions on  $\bar{\Omega}$  that are quasicontinuous on  $\bar{\Omega} \setminus E$  such that the restriction of the function to  $\Omega$  belongs to  $N^{1,p}(\Omega)$ . Such resolutivity result does not follow from the now-classical results in [4].

In the above example we had only one element of the prime end boundary that did not belong to  $\partial_{SP}\Omega$ . We now construct an example where the set  $\partial_P\Omega \setminus \partial_{SP}\Omega$  is uncountable and satisfies  $\bar{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$  while  $C_p(P^{-1}(\partial_P\Omega \setminus \partial_{SP}\Omega)) > 0$ .

**Example 8.3.** In this example we consider a domain in  $\mathbb{R}^3$ :

$$\Omega := (0, 1)^3 \setminus \bigcup_{1 \leq n \in \mathbb{N}} \{1/(2n)\} \times [0, 3/4 + 1/n] \times [0, 1 - 1/n] \cup \{1/(2n+1)\} \times [1/4 - 1/n, 1] \times [1/n, 1].$$

Clearly none of the points in  $E := \{0\} \times [0, 1]^2$  is accessible from  $\Omega$ , and it can be shown using the same technique as in the previous example that  $\bar{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) = 0$ , while  $C_p(E) > 0$ . In this case, note that, for each line segment  $\gamma$  in the 2-dimensional hyperplane region  $E$  that connects the line  $\{0\} \times \{1/4\} \times [0, 1]$  to the line  $\{0\} \times \{3/4\} \times [0, 1]$  and lies in between them, there is a prime end in  $\partial_P\Omega$  with that line as its impression. Such a prime end is obtained by considering acceptable sets  $E_k = \bigcup_{x \in \gamma} B(x, 1/k) \cap \Omega$ . By the construction of  $\Omega$ , it follows that  $E_k$  is connected for each positive integer  $k$ . It follows that  $\partial_P\Omega \setminus \partial_{SP}\Omega$  is uncountable.

**Remark 8.4.** We conclude this section by posing the following two open problems:

- (a) Are there bounded domains that fail the assumption given in Definition 4.7?
- (b) Are there bounded domains for which  $\bar{C}_p^P(\partial_P\Omega \setminus \partial_{SP}\Omega) > 0$ ?

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